A note on quasi-polarized manifolds (X, L) with $h^0(K_X + (n-2)L) = 0$

YOSHIAKI FUKUMA

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Let X be a projective variety of dimension n over the field of complex numbers $\mathbb{C}$, and let L be a nef and big (resp. an ample) line bundle on X. Then we call the pair (X, L) a *quasi-polarized* (resp. *polarized*) *variety*, and (X, L) is called a quasi-polarized (resp. polarized) *manifold* if X is smooth.

In this note, we consider a characterization of quasi-polarized manifolds (X, L) such that L is generated by its global sections and $h^0(K_X + (n-2)L) = 0$.

Definition 1 Let (X, L) be a quasi-polarized variety of dimension n. Then the Euler-Poincaré characteristic $\chi(tL)$ is a polynomial in t of total degree n. We set

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(L) \binom{t+j-1}{j}.$$

Here we will define the ith sectional geometric genus of quasi-polarized varieties.

Definition 2 ([2, Definition 2.1]) Let (X, L) be a quasi-polarized variety of dimension n and let i be an integer with $0 \leq i \leq n$. Then the *ith sectional geometric genus* $g_i(X, L)$ is defined by the following.

$$g_i(X, L) = (-1)^i(\chi_{n-i}(L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X).$$

Remark 1 (1) If $i = 0$, then $g_0(X, L) = L^n$.

(2) If $i = 1$, then $g_1(X, L)$ is the sectional genus $g(X, L)$ of (X, L).

(3) If $i = n$, then $g_n(X, L) = h^n(\mathcal{O}_X)$.

Then by [2, Corollary 3.4] we get the following.

Theorem 1 *Let (X, L) be a quasi-polarized manifold of dimension $n \geq 3$. Assume that L is spanned by its global sections. Then $h^0(K_X + (n-2)L) = 0$ if and only if $g_2(X, L) = h^2(\mathcal{O}_X)$.*

Moreover if L is *ample* and spanned by its global sections, then we get the following by [2, Corollary 3.5].

Theorem 2 *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that L is spanned by its global sections. Then $g_2(X, L) = h^2(\mathcal{O}_X)$ if and only if (X, L) is one of the following types.*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(3) *A scroll over a smooth projective curve.*

(4) $K_X \sim -(n-1)L$*, that is,* (X, L) *is a Del Pezzo manifold.*

(5) *A hyperquadric fibration over a smooth curve.*

(6) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$*, where* S *is a smooth projective surface and* $\mathcal{E}$ *is an ample vector bundle of rank* $n-1$ *on* S*.*

(7) *Let* (M, A) *be a reduction of* (X, L)*.*

(7.1) $n = 4$, $(M, A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.

(7.2) $n = 3$, $(M, A) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.

(7.3) $n = 3$, $(M, A) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.

(7.4) $n = 3$*,* M *is a* $\mathbb{P}^2$*-bundle over a smooth curve* C *and* $(F', A|_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *for any fiber* F' *of it.*

By Theorems 1 and 2, we get a characterization of *polarized* manifolds (X, L) with $h^0(K_X + (n-2)L) = 0$.

Next we consider the case where L is nef and big and spanned by its global sections. Here we use notation and terminologies in [1].

Theorem 3 *Let* (X, L) *be a quasi-polarized manifold of dimension* $n \geq 3$*. Assume that* L *is spanned by its global sections. Then* $h^0(K_X + (n-2)L) = 0$ *if and only if first reduction* (X'', L'') *of* (X, L) *in the sense of* [1, Definition 5.5] *satisfies one of the pairs in* [1, Proposition 3.5, B)].

Proof. (1) First we consider the case where (X, L) satisfies $h^0(K_X + (n-2)L) = 0$. Assume that $K_X + (n-2)L$ is pseudoeffective. Since L is spanned by its global sections, there exist $(n-3)$ members $Y_1, \ldots, Y_{n-3}$ of $|L|$ such that $X_{n-3} := Y_1 \cap \ldots \cap Y_{n-3}$ is smooth. Let $L_{n-3} := L|_{X_{n-3}}$. Then we note that $K_{X_{n-3}} + L_{n-3}$ is pseudoeffective. Hence by [1, Theorem 5.6, 2)] $K_{X''_{n-3}} + L''_{n-3}$ is nef, where (X''_{n-3}, L''_{n-3}) is a first reduction of (X_{n-3}, L_{n-3}) in the sense of [1, Definition 5.5]. Since X''_{n-3} has only $\mathbb{Q}$-factorial terminal singularities, [3, Theorem 1.5] implies that $0 < h^0(K_{X''_{n-3}} + L''_{n-3}) = h^0(K_{X_{n-3}} + L_{n-3})$. Moreover we can easily check that $h^0(K_{X_{n-3}} + L_{n-3}) \leq h^0(K_X + (n-2)L)$. But this contradicts the assumption that $h^0(K_X + (n-2)L) = 0$. Hence $K_X + (n-2)L$ is not pseudoeffective. Therefore by [1, Theorem 5.6, 1)] first reduction (X'', L'') of (X, L) satisfies one of the pairs in [1, Proposition 3.5, B)].
(2) Next we consider the case where first reduction (X'', L'') of (X, L) satisfies one of the pairs in [1, Proposition 3.5, B)]. Then by [1, Theorem 5.6, 1)] we have $h^0(K_X + (n-2)L) = 0$. This completes the proof. □

By Theorems 1 and 3, we also get the following.

Theorem 4 *Let* (X, L) *be a quasi-polarized manifold of dimension* $n \geq 3$*. Assume that* L *is spanned by its global sections. Then* $g_2(X, L) = h^2(\mathcal{O}_X)$ *if and only if first reduction* (X'', L'') *of* (X, L) *in the sense of* [1, Definition 5.5] *satisfies one of the pairs in* [1, Proposition 3.5, B)].

References

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Yoshiaki Fukuma
Department of Mathematics
Faculty of Science
Kochi University
Akebono-cho, Kochi 780-8520
Japan
E-mail: fukuma@kochi-u.ac.jp