On classification of polarized 3-folds (X, L) with $h^0(K_X + 2L) = 2$ *†‡

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Abstract

Let (X, L) be a polarized manifold of dimension $n \geq 3$. In this paper, we give a classification of (X, L) with $n = 3$ and $h^0(K_X + 2L) = 2$. In order to classify these (X, L), we study a classification of (X, L) which satisfies $h^0(K_X + (n-1)L) = 2$ and $\kappa(K_X + (n-2)L) = -\infty$.

1 Introduction

Let (X, L) be a polarized manifold of dimension n. Recently the dimension of global sections of adjoint bundles of (X, L) has been studied actively by several authors (for example [14], [15], [16], [3], [4], [17], [19] and so on). In particular the author has proved the following conjecture for the case of $\dim X = 3$, which was proposed by Beltrametti and Sommese.

Conjecture 1.1 (Beltrametti-Sommese) *Let (X, L) be a polarized manifold of dimension $n \geq 2$. Assume that $K_X + (n-1)L$ is nef. Then $h^0(K_X + (n-1)L) > 0$.*

Here we note that if $K_X + (n-1)L$ is not nef and $n \geq 2$, then (X, L) is one of the following types.

- $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
- $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
- A scroll over a smooth curve.
- $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

In these cases we see that $h^0(K_X + (n-1)L) = 0$. Hence this conjecture says that $h^0(K_X + (n-1)L) = 0$ if and only if (X, L) is one of the above types. In particular, we get a classification of polarized 3-folds (X, L) with $h^0(K_X + 2L) = 0$. Moreover the author gave a classification of polarized 3-folds (X, L) with $h^0(K_X + 2L) = 1$ (see [14, Theorem 2.4]).

In this paper, as the next step, we are going to study polarized 3-folds (X, L) with $h^0(K_X + 2L) = 2$. In this case, we see from [14, Theorem 2.3] that (X, L) satisfies $\kappa(K_X + L) = -\infty$. So in this paper, we will treat more general case, that is, we consider a classification of (X, L) with $\dim X = n \geq 3$, $h^0(K_X + (n-1)L) = 2$ and $\kappa(K_X + (n-2)L) = -\infty$. As a result, we can get a classification of polarized 3-folds (X, L) with $h^0(K_X + 2L) = 2$. Finally, we are going to give some remarks about a classification of (X, L) with $\dim X = n \geq 4$ and $h^0(K_X + (n-1)L) = 2$.

In this paper, we use customary notation in algebraic geometry.

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2 Preliminaries

Definition 2.1 Let (X, L) be a polarized manifold of dimension n. We say that (X, L) is a *scroll* (resp. *quadric fibration*) *over a normal projective variety* Y *of dimension* m with $1 \leq m < n$ if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $K_X + (n-m+1)L = f^*A$ (resp. $K_X + (n-m)L = f^*A$) for some *ample* line bundle A on Y.

Definition 2.2 (1) A polarized manifold is called a *classical scroll over a normal variety* Y if there exists a vector bundle $\mathcal{E}$ on Y such that $X \cong \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$, where $H(\mathcal{E})$ is the tautological line bundle.

(2) We say that a polarized manifold (X, L) is a *hyperquadric fibration over a smooth projective curve* C if (X, L) is a quadric fibration over C and the morphism $f : X \to C$ is the contraction morphism of an extremal ray. In this case, $(F, L_F) \cong (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1))$ for any general fiber F of f and every fiber of f is irreducible and reduced (see [20] or [6, Claim (3.1)]).

Remark 2.1 (1) If (X, L) is a scroll over a normal projective surface S, then S is smooth and (X, L) is also a classical scroll over S (see [2, (3.2.1) Theorem] and [8, Chapter II, (11.8.6)]).

(2) Assume that (X, L) is a quadric fibration over a smooth projective curve C with $\dim X = n \geq 3$. Let $f : X \to C$ be its morphism. By [2, (3.2.6) Theorem] and the proof of [20, Lemma (c) in Section 1], we see that (X, L) is one of the following:

(a) A hyperquadric fibration over a smooth projective curve.

(b) A classical scroll over a smooth projective surface with $\dim X = 3$.

Theorem 2.1 *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Then (X, L) is one of the following types.*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(3) *A scroll over a smooth projective curve.*

(4) $K_X \sim -(n-1)L$*, that is, (X, L) is a Del Pezzo manifold.*

(5) *A hyperquadric fibration over a smooth projective curve.*

(6) *A classical scroll over a smooth projective surface S.*

(7) *Let (X', L') be a reduction of (X, L).*

(7.1) $n = 4$, $(X', L') = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.

(7.2) $n = 3$, $(X', L') = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.

(7.3) $n = 3$, $(X', L') = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.

(7.4) $n = 3$, *X' is a $\mathbb{P}^2$-bundle over a smooth projective curve C such that* $(F', L'|_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *for any fiber F' of it.*

(7.5) $K_{X'} + (n-2)L'$ *is nef.*

Proof. See [1, Proposition 7.2.2, Theorems 7.2.4, 7.3.2 and 7.3.4] and [8, Chapter II, (11.2), (11.7), and (11.8)], or [20, Section 1, Theorem]. □

Remark 2.2 Let (X, L) be a polarized manifold of dimension $n \geq 3$. If (X, L) is one of the types from (1) to (6) in Theorem 2.1, then (X, L) is a reduction of itself.

Remark 2.3 Let (X, L) be a polarized manifold of dimension $n \geq 3$. Then $\kappa(K_X+(n-2)L) = -\infty$ if and only if (X, L) is one of the types from (1) to (7.4) in Theorem 2.1.

Notation 2.1 (See [6, §3].) Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that (X, L) is a hyperquadric fibration over a smooth projective curve C. Let $f : X \to C$ be its morphism. We put $\mathcal{E} := f_*(L)$. Then $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on C. Let $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ be the projective bundle. Then $X \in |2H(\mathcal{E})+\pi^*(B)|$ for some $B \in \mathrm{Pic}(C)$ and $L = H(\mathcal{E})|_X$, where $H(\mathcal{E})$ is the tautological line bundle of $\mathbb{P}_C(\mathcal{E})$. We put $d := L^n$, $e := \deg \mathcal{E}$ and $b := \deg B$.

Remark 2.4 Let (X, L) be a hyperquadric fibration over a smooth projective curve C and we use the notation in Notation 2.1. Then the following hold (see the proof of [14, Theorem 2.2]).

$$\begin{aligned} K_X &= ((-n+1)H(\mathcal{E}) + \pi^*(K_C + \det(\mathcal{E}) + B))|_X \\ &= (-n+1)L + f^*(K_C + \det(\mathcal{E}) + B), \\ g(X, L) &= 2g(C) - 1 + e + b, \\ h^0(K_X + (n-1)L) &= g(X, L) - h^1(\mathcal{O}_X) = g(C) - 1 + e + b. \end{aligned}$$

Proposition 2.1 ([24, Theorem 1.3]) *Let X be a smooth projective surface and let $f : X \to C$ be an elliptic fibration such that $q(X) = g(C) + 1$. Then there exist a smooth projective curve B of genus ≥ 2, a smooth projective curve E of genus 1 and a finite Abelian group $G = \mathbb{Z}_m \times \mathbb{Z}_n$ acting faithfully on B and E and by translations on E such that $X \cong (B \times E)/G$, where G acts diagonally on the product, and $C \cong B/G$. Moreover f is the natural map $(B \times E)/G \to B/G$. Let F be a general fiber of f and let D be a general fiber of $(B \times E)/G \to E/G$. Then $F \cong E$, $D \cong B$ and $DF = \sharp G = mn$.*

Moreover assume that $g(C) = 0$. Let $\mu := \mathrm{lcm}\{m_i\}$ be the least common multiple of the multiplicities of the fibers. Then the group of divisors on X modulo numerical equivalence $Num(X)$ is generated by $(1/\mu)F$ and $(\mu/\gamma)D + (\delta/2\mu)F$, where $\gamma = \sharp G$ and

$$\delta = \begin{cases} 0, & \text{if } (2g(B) - 2)\mu/\gamma \text{ is even} \\ 1, & \text{if } (2g(B) - 2)\mu/\gamma \text{ is odd.} \end{cases}$$

Let $a, b \in \mathbb{Z}$ and we set $N_1 = (1/\mu)F$ and $N_2 = (\mu/\gamma)D + (\delta/2\mu)F$. Then a line bundle on X whose numerical type is $aN_1 + bN_2$ is ample if and only if $2a + \delta b > 0$ and $b > 0$.

3 Main Results

In this section, we are going to study polarized 3-folds (X, L) with $h^0(K_X + 2L) = 2$. As we said in the Introduction, by [14, Theorem 2.3], we see that $\kappa(K_X + L) = -\infty$. So, in order to get a classification of polarized 3-fold (X, L) with $h^0(K_X + 2L) = 2$, we need to study the case where $\kappa(K_X + L) = -\infty$ and $h^0(K_X + 2L) = 2$. Here we consider more general case. Namely we study polarized manifolds (X, L) with $\dim X = n \geq 3$, $\kappa(K_X+(n-2)L) = -\infty$ and $h^0(K_X+(n-1)L) = 2$.

Theorem 3.1 *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that $\kappa(K_X + (n-2)L) = -\infty$. If $h^0(K_X + (n-1)L) = 2$, then (X, L) is one of the following.*

(A) *(X, L) is a hyperquadric fibration over a smooth projective curve C and one of the following holds. (Here we use Notation 2.1.)*

(A.1) *$g(C) = 2$ and d, e, b and n are one of the following types.*

d	e	b	n
3	2	-1	3
2	1	0	≥ 3
1	0	1	≥ 3

(A.2) $g(C) = 1$ *and* d, e *and* b *are one of the following types.*

d	e	b
6	4	-2
5	3	-1
4	2	0
3	1	1
2	0	2
1	-1	3

(A.3) $g(C) = 0$ *and* (X, L) *is one of the types in* [6, (3.30) Theorem].

(B) (X, L) *is a classical scroll over a smooth projective surface* S. *Then there exists an ample vector bundle* $\mathcal{E}$ *on* S *such that* $X = \mathbb{P}_S(\mathcal{E})$ *and* $L = H(\mathcal{E})$, *and* $(S, \mathcal{E})$ *is one of the following.*

(B.1) S *is the Jacobian variety of a smooth projective curve* C *of genus two and* $\mathcal{E} \cong \mathcal{E}_r(C, o) \otimes N$ *for some numerically trivial line bundle* N *on* S, *where* $\mathcal{E}_r(C, o)$ *is the Jacobian bundle of rank* $n - 1$ *for some point* o *on* C.

(B.2) S *is an abelian surface and* $(\det(\mathcal{E}))^2 = 4$.

(B.3) S *is a bielliptic surface and* $(\det(\mathcal{E}))^2 = 4$.

(B.4) $n = 3$, S *is a one point blowing up of* T *and* $\mathcal{E}$ *is an indecomposable ample vector bundle of rank two on* S *with* $\det(\mathcal{E}) = \pi^*(H) - 2E$, *where* T *is either an abelian surface or a bielliptic surface,* $\pi : S \to T$ *is the birational morphism,* E *is the exceptional curve and* H *is an ample line bundle on* T *with* $H^2 = 6$.

(B.5) S *is a minimal surface with* $\kappa(S) = 1$ *and* $\chi(\mathcal{O}_S) = 0$. *Then* X *has an elliptic fibration* $f : S \to C$ *over a smooth projective curve* C. *Let* $m_i F_i$ *(resp.* F*) be a multiple fiber (resp. a general fiber) of* f *and let* t *be the number of multiple fibers. Then* $q(S)$, $g(C)$, t, $(m_1, \ldots, m_t)$, $(\det(\mathcal{E}))F$ *and* $K_S(\det(\mathcal{E}))$ *are one of the types in the Table* 1 *below.*

(B.6) *There exist a smooth projective curve* C *with* $g(C) = 2$ *and vector bundles* $\mathcal{F}$ *and* $\mathcal{G}$ *of rank two on* C *such that* $\mathcal{F}$ *is normalized,* $S \cong \mathbb{P}_C(\mathcal{F})$ *and* $\mathcal{E} \cong h^*(\mathcal{G}) \otimes H(\mathcal{F})$, *where* $h : S \to C$ *is the projection. Then* $\det \mathcal{E} \equiv 2C_0 + (1 - \deg \mathcal{F})F$ *and* $(\deg \mathcal{F}, \deg \mathcal{G}) = (0, 1), (1, 0), (2, -1)$, *where* C_0 *is the minimal section of* $\mathbb{P}_C(\mathcal{F}) \to C$. *In particular the rank of* $\mathcal{E}$ *is two and* $n = 3$. *Moreover if* $\deg \mathcal{F} \geq 1$, *then* $\mathcal{F}$ *and* $\mathcal{G}$ *are semistable.*

(B.7) $(S, \mathcal{E})$ *is one of the types in* [21, (2.3) Theorem (V)].

(B.8) $(S, \mathcal{E})$ *is either* 4), $5)_0$ *or* $5)_1$ *in* [7, (2.25) Theorem].

(C) *Let* (M, A) *be a reduction of* (X, L). *Then* (M, A) *is a* $\mathbb{P}^2$*-bundle* $\Phi : M \to C$ *over a smooth elliptic curve* C *with* $A|_F \cong \mathcal{O}_{\mathbb{P}^2}(2)$ *for every fiber* F *of* Φ. *In this case, there exists a stable vector bundle* $\mathcal{E}$ *of rank three on* C *with* $c_1(\mathcal{E}) = 2$ *such that* $M \cong \mathbb{P}_C(\mathcal{E})$ *and* $A = 2H(\mathcal{E}) + \Phi^*(B)$ *for some line bundle* B *on* C *with* $\det(\mathcal{E}) + 2B = 0$.

Proof. If $K_X + (n-1)L$ is not nef, then (X, L) is either (1), (2) or (3) in Theorem 2.1. Then we see that $h^0(K_X + (n-1)L) = 0$. So we may assume that $K_X + (n-1)L$ is nef. If $\kappa(K_X + (n-2)L) = -\infty$, then by assumption, Theorem 2.1, and Remark 2.3, (X, L) is one of the following types.

(a) $K_X \sim -(n-1)L$, that is, (X, L) is a Del Pezzo manifold.

(b) A hyperquadric fibration over a smooth projective curve.

(c) A classical scroll over a smooth projective surface.

(d) Let (M, A) be a reduction of (X, L).

Table 1: The list of the possible cases of (B.5) in Theorem 3.1

	$q(S)$	$g(C)$	t	$(m_1, \ldots, m_t)$	$(\det(\mathcal{E}))F$	$K_S(\det(\mathcal{E}))$
(1)	2	1	2	$(2,2)$	2	2
(2)	1	0	6	$(2,2,2,2,2,2)$	2	2
(3)	1	0	5	$(2,2,2,2,2)$	2	1
(4)	1	0	4	$(3,3,3,3)$	3	2
(5)	1	0	4	$(6,2,2,2)$	6	2
(6)	1	0	4	$(4,2,2,2)$	4	1
(7)	1	0	4	$(3,2,2,2)$	6	1
(8)	1	0	4	$(4,4,2,2)$	4	2
(9)	1	0	4	$(3,3,2,2)$	6	2
(10)	1	0	3	$(5,5,5)$	5	2
(11)	1	0	3	$(4,4,4)$	4	1
(12)	1	0	3	$(6,6,3)$	6	2
(13)	1	0	3	$(6,3,3)$	6	1
(14)	1	0	3	$(4,4,3)$	12	2
(15)	1	0	3	$(4,3,3)$	12	1
(16)	1	0	3	$(9,3,3)$	9	2
(17)	1	0	3	$(5,3,3)$	15	2
(18)	1	0	3	$(10,3,2)$	30	2
(19)	1	0	3	$(18,3,2)$	18	2
(20)	1	0	3	$(12,4,2)$	12	2
(21)	1	0	3	$(8,8,2)$	8	2
(22)	1	0	3	$(10,5,2)$	10	2
(23)	1	0	3	$(7,3,2)$	42	1
(24)	1	0	3	$(8,3,2)$	24	1
(25)	1	0	3	$(9,3,2)$	18	1
(26)	1	0	3	$(12,3,2)$	12	1
(27)	1	0	3	$(5,4,2)$	20	1
(28)	1	0	3	$(6,4,2)$	12	1
(29)	1	0	3	$(8,4,2)$	8	1
(30)	1	0	3	$(6,6,2)$	6	1
(31)	1	0	3	$(5,5,2)$	10	1
(32)	2	2	0	nothing	1	2
(33)	1	1	1	(2)	2	1
(34)	1	1	2	$(2,2)$	2	2
(35)	1	1	1	(2)	4	2
(36)	1	1	1	(3)	3	2

(d.1) $n = 4$, $(M, A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.
(d.2) $n = 3$, $(M, A) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.
(d.3) $n = 3$, $(M, A) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.
(d.4) $n = 3$, M is a $\mathbb{P}^2$-bundle over a smooth projective curve C and for any fiber F of it, $(F, A|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

Note that since $h^0(K_X+(n-1)L) = h^0(K_M+(n-1)A)$, we may assume that $(X, L) = (M, A)$.

(a) The case in which (X, L) is a Del Pezzo manifold.
Then $\mathcal{O}_X(K_X+(n-1)L) \sim \mathcal{O}_X$ and $h^0(K_X+(n-1)L) = 1$. Hence this case cannot occur because we assume that $h^0(K_X + (n-1)L) = 2$.

(b) The case in which (X, L) is a hyperquadric fibration over a smooth projective curve C.
Here we use Notation 2.1. Here we note that $e+b > 0$ by [14, Claim 2.1] and $s := 2e+(n+1)b \geq 0$ by [6, (3.3)]. We also note that $g(X, L) = 2g(C)-1+e+b$, $d = L^n = 2e+b$ and $h^0(K_X+(n-1)L) = g(C) - 1 + e + b$ (see Remark 2.4). Hence we get the following type.

(α) $g(C) = 2$, $e + b = 1$, and $g(X, L) = 4$.

(β) $g(C) = 1$, $e + b = 2$, and $g(X, L) = 3$.

(γ) $g(C) = 0$, $e + b = 3$, and $g(X, L) = 2$.

(b.1) First we consider the case (α). Then $d = 2e+b = 2-b$ and $s = 2e+(n+1)b = 2n+(-n+1)d$. Since $s \geq 0$, we have $(n-1)d \leq 2n$, that is

$$d \leq 2 + \frac{2}{n-1}.$$

Hence $d \leq 3$ and if $d = 3$, then $n = 3$. Therefore we get the list in (A.1) of Theorem 3.1.

(b.2) Next we consider the case (β). Then by [22, (2.25) Theorem] we have $1 \leq d \leq 6$. Since $d = 2e + b$ and $e + b = 2$, we have $e = d - 2$ and $b = 4 - d$. So we get the list in (A.2) of Theorem 3.1.

(b.3) Finally we consider the case (γ). Then $g(X, L) = 2$ and we can use Fujita's classification of (X, L) with $g(X, L) = 2$. There are 11 types in this case. For detail, see [6, (3.30) Theorem]. This is the case (A.3) of Theorem 3.1.

(c) The case in which (X, L) is a classical scroll over a smooth projective surface S.
Let $\pi : X \to S$ be the $\mathbb{P}^{n-2}$-bundle over S. Then there exists a vector bundle $\mathcal{E}$ of rank $n-1$ on S such that $X = \mathbb{P}_S(\mathcal{E})$ and $L = H(\mathcal{E})$, where $H(\mathcal{E})$ is the tautological line bundle of $\mathbb{P}_S(\mathcal{E})$. Then $\mathcal{E}$ is ample. By the canonical bundle formula, $K_X = \pi^*(K_S + \det(\mathcal{E})) - (n-1)H(\mathcal{E})$. Hence $K_X + (n-1)L = \pi^*(K_S + \det(\mathcal{E}))$ and we have

$$\begin{aligned} h^0(K_X + (n-1)L) &= h^0(\pi^*(K_S + \det(\mathcal{E}))) \\ &= h^0(K_S + \det(\mathcal{E})) \\ &= g(S, \det(\mathcal{E})) - q(S) + p_g(S). \end{aligned}$$

Since $h^0(K_X + (n-1)L) = 2 > 0$, we see that $K_X + (n-1)L$ is nef by [8, Theprems (11.2) and (11.7)] or [20, Theorem in Section 1]. Hence $K_S + \det(\mathcal{E})$ is nef. Since $0 < L^n = H(\mathcal{E})^n = c_1(\mathcal{E})^2 - c_2(\mathcal{E})$ and $c_2(\mathcal{E}) > 0$, we see that

$$c_1(\mathcal{E})^2 = L^n + c_2(\mathcal{E}) \geq 2. \quad (1)$$

(c.1) Assume that $\kappa(S) \geq 0$. Then $\chi(\mathcal{O}_S) \geq 0$, that is, $h^2(\mathcal{O}_S) - h^1(\mathcal{O}_S) \geq -1$. Hence $h^0(K_X + (n-1)L) \geq g(S, \det(\mathcal{E})) - 1$, and $g(S, \det(\mathcal{E})) \leq 3$ because $h^0(K_X + (n-1)L) = 2$.

(c.1.a) We assume that $\chi(\mathcal{O}_S) \geq 1$. Then $g(S, \det\mathcal{E}) \leq 2$.

(c.1.a.1) If $g(S, \det\mathcal{E}) \leq 1$, then $\kappa(S) = -\infty$ by [7, Theorems (1.4) and (1.5)] and this is impossible.

(c.1.a.2) If $g(S, \det\mathcal{E}) = 2$, then $(S, \mathcal{E})$ is the case (B.1) in Theorem 3.1 by [7, (2.25) Theorem].

(c.1.b) We assume that $\chi(\mathcal{O}_S) = 0$. Then $\kappa(S) = 0$ or 1. On the other hand $g(S, \det(\mathcal{E})) = 3$ since $h^0(K_X + (n-1)L) = 2$. So by [9, Theorem 2.1] we have $q(S) \leq 3$.

(c.1.b.1) Assume that $q(S) = 3$. Then by [9, Theorem 3.1], we see that $\kappa(S) = 1$ and $(S, \det(\mathcal{E})) \cong (E_1 \times E_2, E_1 + E_2)$, where E_1 and E_2 are smooth projective curves with $g(E_1) = 1$ and $g(E_2) = 2$. Let $p_2 : S \to E_2$ be the second projection. Then by [7, (2.14) Lemma] we have $(p_2)_*(\mathcal{E})$ is a line bundle on E_2 with $\delta = \deg((p_2)_*(\mathcal{E})) > 0$ and $c_1(\mathcal{E})^2 = 2r\delta$, where $r = \mathrm{rank}(\mathcal{E})$. But since $c_1(\mathcal{E})^2 = (E_1 + E_2)^2 = 2$ and $r \geq 2$, this is impossible.

(c.1.b.2) Assume that $q(S) \leq 2$.

(c.1.b.2.1) If $\kappa(S) = 0$, then S is birationally equivalent to an abelian surface or a bielliptic surface because $\chi(\mathcal{O}_S) = 0$.

(c.1.b.2.1.1) Assume that S is minimal. Since $K_S \det(\mathcal{E}) = 0$, we have $(\det\mathcal{E})^2 = 4$. Then $(S, \mathcal{E})$ is either the case (B.2) or the case (B.3) in Theorem 3.1.

(c.1.b.2.1.2) Assume that S is not minimal. Then by [21, (2.3) Theorem], S is a one point blowing up of a smooth projective surface S' such that S' is minimal and $\det(\mathcal{E}) = \pi^*(H) - 2E$, where $\pi : S \to S'$ is the blowing up, E is the exceptional divisor and H is an ample line bundle on S'. Then $H^2 = 6$. Since $\mathrm{rank}(\mathcal{E}) \leq \det(\mathcal{E})E = 2$ we have $\mathrm{rank}(\mathcal{E}) = 2$, that is, $n = 3$ in this case. Moreover the following holds.

Claim 3.1 *$\mathcal{E}$ is indecomposable.*

Proof. Assume that $\mathcal{E}$ is decomposable. We set $\mathcal{E} = L_1 \oplus L_2$. Since $\mathcal{E}$ is ample, so are L_1 and L_2. Moreover $L_1 + L_2 = \pi^*(H) - 2E$. Let $A_i := \pi_*(L_i)$ for $i = 1, 2$. Then each A_i is ample and $H = A_1 + A_2$. Since $\kappa(S') = 0$ and S' is minimal, we have $K_{S'} A_i = 0$ for $i = 1, 2$. Therefore A_i^2 is a positive even number. Hence by the Hodge index theorem, we have $A_1 A_2 \geq 2$. Therefore $(H)^2 \geq 8$ and this is a contradiction because $(H)^2 = 6$. This completes the proof. □

This $(S, \mathcal{E})$ is the case (B.4) in Theorem 3.1.

(c.1.b.2.2) If $\kappa(S) = 1$, then by [21, (2.3) Theorem] we see that S is minimal. By the classification theory of surfaces, there exists an elliptic fibration $f : S \to C$ over a smooth projective curve C. Then $q(S) = g(C)$ or $q(S) = g(C) + 1$. By the canonical bundle formula for elliptic fibrations, we have $K_S = f^*(B) + \sum_i (m_i - 1)F_i$ for some line bundle B on C with $\deg B = \chi(\mathcal{O}_S) - 2\chi(\mathcal{O}_C) = 2g(C) - 2 + \chi(\mathcal{O}_S)$, where $m_i F_i$ is a multiple fiber of f.

(c.1.b.2.2.1) First we consider the case where $q(S) = g(C) + 1$. Then f has no multiple fiber or at least two multiple fibers (see [29, Lemma 1.6] and [30, Proposition 1.3]). On the other hand since $\kappa(S) = 1$, we have $K_S(\det(\mathcal{E})) > 0$. As we said above (see (1)), $(\det(\mathcal{E}))^2 = c_1(\mathcal{E})^2 \geq 2$ holds. Since $g(S, \det(\mathcal{E})) = 3$, we get

$$K_S(\det(\mathcal{E})) = 1 \text{ or } 2. \tag{2}$$

(i) The case where $q(S) = 2$ and $g(C) = 1$.
Then $K_S \equiv \sum_i (m_i - 1)F_i$ because $\chi(\mathcal{O}_S) = 0$. Hence f has at least two multiple fibers and we get $K_S(\det(\mathcal{E})) \geq 2$. Therefore by (2) we have $K_S(\det(\mathcal{E})) = 2$ and f has just two multiple fibers and multiple fibers are $2F_1$ and $2F_2$. Moreover $(\det(\mathcal{E}))F_i = 1$ for $i = 1, 2$. This is the type (1) in Table 1.

(ii) The case where $q(S) = 1$ and $g(C) = 0$.
Since $\chi(\mathcal{O}_S) = 0$ by assumption (see (c.1.b)), we have $K_S \equiv -2F + \sum_i (m_i - 1)F_i$, where F is a general fiber of f. We see from (2) that $K_S(\det(\mathcal{E})) = 1$ or 2 holds. Since $K_S(\det(\mathcal{E})) \geq 1$, we see that f has at least three multiple fibers. Here we may assume that $m_1 \geq m_2 \geq \cdots$ hold. Since $m_i F_i \equiv m_j F_j$, we have

$$F_i \det(\mathcal{E}) \leq F_j \det(\mathcal{E}) \qquad \text{for every } i < j. \tag{3}$$

We note that $F \equiv m_1 F_1 \equiv m_2 F_2$. Hence

$$\begin{aligned} K_S \det(\mathcal{E}) &= \left(\sum_i (m_i - 1)F_i - 2F \right) \det(\mathcal{E}) \\ &= \left(-F_1 - F_2 + \sum_{i \geq 3} (m_i - 1)F_i \right) \det(\mathcal{E}). \end{aligned} \tag{4}$$

(ii.1) Assme that the number of multiple fibers is three. Then we see from (4) that $K_S \det(\mathcal{E}) = (-F_1 - F_2 + (m_3 - 1)F_3) \det(\mathcal{E})$. Since $K_S(\det(\mathcal{E})) \leq 2$, we have $m_3 \leq 5$ by (3).

(ii.1.1) If $m_3 = 5$, then $F_1 \det(\mathcal{E}) = F_2 \det(\mathcal{E}) = F_3 \det(\mathcal{E}) = 1$ and $m_1 = m_2 = 5$. Hence $\det(\mathcal{E})F = 5$. This is the type (10) in Table 1.

(ii.1.2) If $m_3 = 4$, then $F_3 \det(\mathcal{E}) = 2$ or 1.

(ii.1.2.1) If $F_3 \det(\mathcal{E}) = 2$, then $\det(\mathcal{E})F = 8$. Since $K_S \det(\mathcal{E}) \leq 2$, we have $F_1 \det(\mathcal{E}) = F_2 \det(\mathcal{E}) = 2$ and $m_1 = m_2 = 4$. In this case $K_S \det(\mathcal{E}) = 2$.

Claim 3.2 *This case cannot occur.*

Proof. We use Proposition 2.1 and the notation in Proposition 2.1. Let $\det(\mathcal{E}) = \alpha N_1 + \beta N_2$, and $K_S \equiv \epsilon F$, where $\alpha, \beta \in \mathbb{Z}$ and $\epsilon \in \mathbb{Q}$. Then

$$K_S(\det(\mathcal{E})) = \epsilon\mu\beta, \tag{5}$$

$$(\det(\mathcal{E}))^2 = \beta(2\alpha + \beta\delta). \tag{6}$$

Here we prove the assertion of this claim. In this case, we see from (5) that $\mu = 4$, $\epsilon = \frac{1}{4}$ and $\beta = 2$ since $K_S(\det(\mathcal{E})) = 2$. Hence by (6) we have $2 = (\det(\mathcal{E}))^2 = 2(2\alpha + 2\delta)$. But this is imposible because α and δ are integer. This completes the proof of this claim. $\square$

(ii.1.2.2) If $F_3 \det(\mathcal{E}) = 1$, then $\det(\mathcal{E})F = 4$. By (3) we have $F_1(\det \mathcal{E}) = F_2 \det(\mathcal{E}) = 1$ and $m_1 = m_2 = 4$. In this case $K_S \det(\mathcal{E}) = 1$. This is the case (11) in Table 1.

(ii.1.3) If $m_3 = 3$, then $0 \leq F_3 \det(\mathcal{E}) - F_1 \det(\mathcal{E}) \leq 2$.

(ii.1.3.1) If $F_3 \det(\mathcal{E}) - F_1 \det(\mathcal{E}) = 0$, then we see from (3) that $F_3 \det(\mathcal{E}) = F_2 \det(\mathcal{E}) = F_1 \det(\mathcal{E})$. But in this case $K_S \det(\mathcal{E}) = 0$ and this is impossible because $\kappa(S) = 1$ and $\det(\mathcal{E})$ is ample.

(ii.1.3.2) If $F_3 \det(\mathcal{E}) - F_1 \det(\mathcal{E}) = 1$, then $F_3 \det(\mathcal{E}) - F_2 \det(\mathcal{E}) = 0$ or 1. Since

$$\begin{aligned} F_1\det(\mathcal{E}) &= \frac{3}{m_1}F_3\det(\mathcal{E}) \\ F_2\det(\mathcal{E}) &= \frac{3}{m_2}F_3\det(\mathcal{E}), \end{aligned}$$

we have $1 = F_3\det(\mathcal{E}) - F_1\det(\mathcal{E}) = \left(\frac{m_1-3}{m_1}\right) F_3\det(\mathcal{E})$, that is,

$$F_3\det(\mathcal{E}) = 1 + \frac{3}{m_1 - 3}.$$

Therefore $m_1 = 4$ or 6.

If $F_3\det(\mathcal{E}) - F_2\det(\mathcal{E}) = 0$, then $m_2 = m_3 = 3$.

If $F_3\det(\mathcal{E}) - F_2\det(\mathcal{E}) = 1$, then we have $1 = F_3\det(\mathcal{E}) - F_2\det(\mathcal{E}) = \left(\frac{m_2-3}{m_2}\right) F_3\det(\mathcal{E})$, that is,

$$F_3\det(\mathcal{E}) = 1 + \frac{3}{m_2 - 3}.$$

Hence $m_2 = 4$ or 6. Therefore we get the following types.

m_1	m_2	m_3	$F_1 \det(\mathcal{E})$	$F_2 \det(\mathcal{E})$	$F_3 \det(\mathcal{E})$
6	6	3	1	1	2
6	3	3	1	2	2
4	4	3	3	3	4
4	3	3	3	4	4

These are the cases (12), (13), (14) and (15) in Table 1.

(ii.1.3.3) If $F_3 \det(\mathcal{E}) - F_1 \det(\mathcal{E}) = 2$, then we see that $F_3 \det(\mathcal{E}) = F_2 \det(\mathcal{E})$. Since $F_1\det(\mathcal{E}) = \frac{3}{m_1}F_3\det(\mathcal{E})$, we have $2 = F_3\det(\mathcal{E}) - F_1\det(\mathcal{E}) = \left(\frac{m_1-3}{m_1}\right) F_3\det(\mathcal{E})$, that is,

$$F_3\det(\mathcal{E}) = 2 + \frac{6}{m_1 - 3}.$$

Hence $m_1 = 4$, 5, 6 or 9. Therefore we obtain the following table.

	m_1	m_2	m_3	$F_1 \det(\mathcal{E})$	$F_2 \det(\mathcal{E})$	$F_3 \det(\mathcal{E})$
(a)	9	3	3	1	3	3
(b)	6	3	3	2	4	4
(c)	5	3	3	3	5	5
(d)	4	3	3	6	8	8

By the same argument as Claim 3.2, we see that the cases (b) and (d) cannot occur. The case (a) (resp. (c)) is the case (16) (resp. (17)) in Table 1.

(ii.1.4) If $m_3 = 2$, then $K_S \det(\mathcal{E}) = (-F_1 - F_2 + F_3) \det(\mathcal{E})$. In this case we have

$$F_3 \det(\mathcal{E}) > F_2 \det(\mathcal{E}) \tag{7}$$

(ii.1.4.1) Assume that $K_S \det(\mathcal{E}) = 2$. Then

$$\det(\mathcal{E})F_3 = \det(\mathcal{E})F_1 + \det(\mathcal{E})F_2 + 2. \tag{8}$$

Since $m_1F_1 \equiv m_2F_2 \equiv 2F_3$, we have

$$\det(\mathcal{E})F_1 = \frac{2}{m_1}\det(\mathcal{E})F_3, \qquad \det(\mathcal{E})F_2 = \frac{2}{m_2}\det(\mathcal{E})F_3.$$

Hence we get

$$\det(\mathcal{E})F_3 = \frac{2m_1m_2}{(m_1-2)(m_2-2)-4}. \tag{9}$$

Here we note that $m_2 \geq 3$ because $\det(\mathcal{E})F_3 > \det(\mathcal{E})F_2$ by (7).

(ii.1.4.1.1) If $m_2 = 3$, then by (9)

$$\det(\mathcal{E})F_3 = 6 + \frac{36}{m_1 - 6}.$$

In this case, since

$$\det(\mathcal{E})F_1 = \frac{2}{m_1}\det(\mathcal{E})F_3 = \frac{12}{m_1-6},$$

the following are possible.

	m_1	$\det(\mathcal{E})F_1$	$\det(\mathcal{E})F_2$	$\det(\mathcal{E})F_3$
(a)	7	12	28	42
(b)	8	6	16	24
(c)	9	4	12	18
(d)	10	3	10	15
(e)	12	2	8	12
(f)	18	1	6	9

By the same argument as Claim 3.2, we see that the cases (a), (b), (c) and (e) cannot occur. The case (d) (resp. (f)) is the case (18) (resp. (19)) in Table 1.

(ii.1.4.1.2) If $m_2 = 4$, then by (9)

$$\det(\mathcal{E})F_3 = 4 + \frac{32}{2m_1 - 8}.$$

In this case, the following are possible.

	m_1	$\det(\mathcal{E})F_1$	$\det(\mathcal{E})F_2$	$\det(\mathcal{E})F_3$
(a)	5	8	10	20
(b)	6	4	6	12
(c)	8	2	4	8
(d)	12	1	3	6

By the same argument as Claim 3.2, we see that the cases (a), (b) and (c) cannot occur. The case (d) is the case (20) in Table 1.

(ii.1.4.1.3) Assume that $m_2 \geq 5$. Then $m_1 \geq 5$. First we have

$$m_1\det(\mathcal{E})F_1 + m_2\det(\mathcal{E})F_2 = 2\det(\mathcal{E})F_3 + 2\det(\mathcal{E})F_3 = 4\det(\mathcal{E})F_3 = 4\det(\mathcal{E})F_1 + 4\det(\mathcal{E})F_2 + 8.$$

(Note that we use (8) at the last equality.) Namely

$$(m_1 - 4)\det(\mathcal{E})F_1 + (m_2 - 4)\det(\mathcal{E})F_2 = 8. \tag{10}$$

Hence

$$\det(\mathcal{E})F_1 + \det(\mathcal{E})F_2 \le (m_1 - 4)\det(\mathcal{E})F_1 + (m_2 - 4)\det(\mathcal{E})F_2 = 8.$$

Therefore $(\det(\mathcal{E})F_1, \det(\mathcal{E})F_2) = (1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(1,7)$, $(2,2)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(3,3)$, $(3,4)$, $(3,5)$ or $(4,4)$. Then we get the following possiblities by using (10).

	m_1	m_2	$\det(\mathcal{E})F_1$	$\det(\mathcal{E})F_2$	$\det(\mathcal{E})F_3$
(a)	8	8	1	1	4
(b)	10	5	1	2	5
(c)	6	6	2	2	6
(d)	5	5	4	4	10

By the same argument as Claim 3.2, we see that the cases (c) and (d) cannot occur. The case (a) (resp. (b)) is the case (21) (resp. (22)) in Table 1.

(ii.1.4.2) Assume that $K_S \det(\mathcal{E}) = 1$. Then

$$\det(\mathcal{E})F_3 = \det(\mathcal{E})F_2 + \det(\mathcal{E})F_1 + 1. \tag{11}$$

Hence we find that

$$\begin{aligned} m_1 \det(\mathcal{E})F_1 &= 2\det(\mathcal{E})F_3 = 2\det(\mathcal{E})F_2 + 2\det(\mathcal{E})F_1 + 2, &(12)\\ m_2 \det(\mathcal{E})F_2 &= 2\det(\mathcal{E})F_3 = 2\det(\mathcal{E})F_2 + 2\det(\mathcal{E})F_1 + 2. &(13)\end{aligned}$$

On the other hand, since $\det(\mathcal{E})F_1 = (2/m_1)\det(\mathcal{E})F_3$ and $\det(\mathcal{E})F_2 = (2/m_2)\det(\mathcal{E})F_3$, we see from (11) that

$$\det(\mathcal{E})F_3 = (2/m_1)\det(\mathcal{E})F_3 + (2/m_2)\det(\mathcal{E})F_3 + 1.$$

Therefore

$$\left(1 - \frac{2}{m_1} - \frac{2}{m_2}\right)\det(\mathcal{E})F_3 = 1,$$

that is,

$$\det(\mathcal{E})F_3 = \frac{m_1 m_2}{(m_1 - 2)(m_2 - 2) - 4}.$$

Here we note that $m_2 \ge 3$ because $m_3 = 2$ and $\det(\mathcal{E})F_3 > \det(\mathcal{E})F_2$ by (7).

(ii.1.4.2.1) First we consider the case in which $m_2 = 3$. Then

$$\det(\mathcal{E})F_3 = \frac{3m_1}{m_1 - 6} = 3 + \frac{18}{m_1 - 6}.$$

Then the following are possible.

m_1	$\det(\mathcal{E})F_1$	$\det(\mathcal{E})F_2$	$\det(\mathcal{E})F_3$
7	6	14	21
8	3	8	12
9	2	6	9
12	1	4	6

These are the cases (23), (24), (25) and (26) in Table 1.

(ii.1.4.2.2) Next we consider the case in which $m_2 = 4$. Here we note that $m_1 \geq 4$. In this case we have

$$\det(\mathcal{E})F_3 = \frac{2m_1}{m_1 - 4} = 2 + \frac{8}{m_1 - 4}.$$

Then we get the following possible type.

m_1	$\det(\mathcal{E})F_1$	$\det(\mathcal{E})F_2$	$\det(\mathcal{E})F_3$
5	4	5	10
6	2	3	6
8	1	2	4

These are the cases (27), (28) and (29) in Table 1.

(ii.1.4.2.3) Finally we consider the case in which $m_2 \geq 5$. Then $m_1 \geq 5$, and since $K_X \det(\mathcal{E}) = 1$ and $m_3 = 2$, we see from (12) and (13) that

$$\det(\mathcal{E})F_1 + \det(\mathcal{E})F_2 \leq (m_1 - 4)\det(\mathcal{E})F_1 + (m_2 - 4)\det(\mathcal{E})F_2 = 4.$$

Therefore $(\det(\mathcal{E})F_1, \det(\mathcal{E})F_2) = (1,1), (1,2), (1,3), (2,2)$. Since $\det(\mathcal{E})F_3 = \det(\mathcal{E})F_1 + \det(\mathcal{E})F_2 + 1$, $(m_1 - 4)\det(\mathcal{E})F_1 + (m_2 - 4)\det(\mathcal{E})F_2 = 4$ and $m_3 = 2$, we get the following:

m_1	m_2	$\det(\mathcal{E})F_1$	$\det(\mathcal{E})F_2$	$\det(\mathcal{E})F_3$
6	6	1	1	3
5	5	2	2	5

These are the cases (30) and (31) in Table 1.

(ii.2) Assme that the number of multiple fibers is four. Then $K_S \det(\mathcal{E}) = (-F_1 - F_2 + (m_3 - 1)F_3 + (m_4 - 1)F_4)\det(\mathcal{E})$. Since $1 \leq K_S \det(\mathcal{E}) \leq 2$, we have $(m_3, m_4) = (4,2), (3,3), (3,2), (2.2)$.

(ii.2.1) If $(m_3, m_4) = (4,2)$, then $K_S \det(\mathcal{E}) = (-F_1 - F_2 + 3F_3 + F_4)\det(\mathcal{E}) = 2\det(\mathcal{E})F_3 + (\det(\mathcal{E})F_3 - \det(\mathcal{E})F_1) + (\det(\mathcal{E})F_4 - \det(\mathcal{E})F_2) \geq 2\det(\mathcal{E})F_3 \geq 2$. Since $K_S \det(\mathcal{E}) \leq 2$, we have $\det(\mathcal{E})F_3 = 1$, $\det(\mathcal{E})F_3 = \det(\mathcal{E})F_1$ and $\det(\mathcal{E})F_4 = \det(\mathcal{E})F_2$. Therefore $\det(\mathcal{E})F_1 = \det(\mathcal{E})F_2 = \det(\mathcal{E})F_3 = \det(\mathcal{E})F_4 = 1$. But this is impossible.

(ii.2.2) If $(m_3, m_4) = (3,3)$, then the following holds by the same argument as in (ii.2.1).

m_1	m_2	m_3	m_4	$K_S \det(\mathcal{E})$	$\det(\mathcal{E})F_1$	$\det(\mathcal{E})F_2$	$\det(\mathcal{E})F_3$	$\det(\mathcal{E})F_4$
3	3	3	3	2	1	1	1	1

This is the case (4) in Table 1.

(ii.2.3) Next we consider the case where $(m_3, m_4) = (3,2)$.

(ii.2.3.1) If $(m_3, m_4) = (3,2)$ and $K_S \det(\mathcal{E}) = 1$, then we see that $\det(\mathcal{E})F_i = 1$ for $i = 1,2,3,4$. But since $m_3 \neq m_4$ this case cannot occur.

(ii.2.3.2) If $(m_3, m_4) = (3,2)$ and $K_S \det(\mathcal{E}) = 2$, then we see that $\det(\mathcal{E})F_4 = 1$ or 2 because $2\det(\mathcal{E})F_3 + \det(\mathcal{E})F_4 - \det(\mathcal{E})F_1 - \det(\mathcal{E})F_2 = 2$.

(ii.2.3.2.1) If $\det(\mathcal{E})F_4 = 2$, then $\det(\mathcal{E})F = 4$ because $m_4 = 2$. But this is impossible because

$m_3 = 3$.

(ii.2.3.2.2) If $\det(\mathcal{E})F_4 = 1$, then $\det(\mathcal{E})F_3 = \det(\mathcal{E})F_2 = \det(\mathcal{E})F_1 = 1$ because $\det(\mathcal{E})F_i \geq \det(\mathcal{E})F_{i-1}$. But this is impossible because $m_3 \neq m_4$.

(ii.2.4) Assume that $(m_3, m_4) = (2, 2)$. Then $K_S \det(\mathcal{E}) = (-F_1 - F_2 + F_3 + F_4)\det(\mathcal{E}) = (F_4 - F_1)\det(\mathcal{E}) + (F_3 - F_2)\det(\mathcal{E})$. Since $1 \leq K_S \det(\mathcal{E}) \leq 2$, we see from (3) that

$$((F_4 - F_1)\det(\mathcal{E}), (F_3 - F_2)\det(\mathcal{E})) = (2, 0), (1, 1), (1, 0).$$

(ii.2.4.1) If $(F_3 - F_2)\det(\mathcal{E}) = 0$, then $(F_2 - F_1)\det(\mathcal{E}) > 0$ and in this case we have the following table.

	m_1	m_2	m_3	m_4	$K_S \det(\mathcal{E})$	$\det(\mathcal{E})F_1$	$\det(\mathcal{E})F_2$	$\det(\mathcal{E})F_3$	$\det(\mathcal{E})F_4$
(a)	6	2	2	2	2	1	3	3	3
(b)	4	2	2	2	2	2	4	4	4
(c)	3	2	2	2	2	4	6	6	6
(d)	4	2	2	2	1	1	2	2	2
(e)	3	2	2	2	1	2	3	3	3

By the same argument as Claim 3.2, we see that the cases (b) and (c) cannot occur. The case (a) (resp. (d), (e)) is the case (5) (resp. (6) and (7)) in Table 1.

(ii.2.4.2) If $(F_3 - F_2)\det(\mathcal{E}) = 1$, then $(F_3 - F_2)\det(\mathcal{E}) > 0$ and $(F_2 - F_1)\det(\mathcal{E}) = 0$. Hence in this case we have the following table.

m_1	m_2	m_3	m_4	$K_S \det(\mathcal{E})$	$\det(\mathcal{E})F_1$	$\det(\mathcal{E})F_2$	$\det(\mathcal{E})F_3$	$\det(\mathcal{E})F_4$
4	4	2	2	2	1	1	2	2
3	3	2	2	2	2	2	3	3

These are the cases (8) and (9) in Table 1.

(ii.3) Assume that the number of multiple fibers is greater than or equal to five.
First we see from (3) that we have

$$K_S \det(\mathcal{E}) \quad \geq \quad \left((m_3 - 2)F_3 + (m_4 - 2)F_4 + \sum_{i \geq 5} (m_i - 1)F_i \right) \det(\mathcal{E}). \tag{14}$$

(ii.3.1) The case where $K_S(\det(\mathcal{E})) = 2$.
Then by (14) we get $m_5 \leq 3$. Assume that $m_5 = 3$. Then since $m_1 \geq m_2 \geq \cdots$ hold, we see that $m_i \geq 3$ for $i = 1, 2, 3, 4$, and by (14) we have

$$K_S(\det(\mathcal{E})) \geq \left(F_3 + F_4 + \sum_{i \geq 5} (m_i - 1)F_i \right) \det(\mathcal{E}) \geq 4$$

and this is a contradiction. Hence $m_5 = 2$, and since $K_S(\det(\mathcal{E})) \geq F_5 \det(\mathcal{E})$, we have $\det(\mathcal{E})F_5 = 2$ or 1.

(ii.3.1.1) If $\det(\mathcal{E})F_5 = 2$, then the equality in (14) holds. So $m_3 = 2$, $m_4 = 2$ and $m_5 = 2$, and $\det(\mathcal{E})F_1 = \det(\mathcal{E})F_2 = \det(\mathcal{E})F_3 = \det(\mathcal{E})F_4$. Hence $m_i = 2$ and $(\det(\mathcal{E}))F_i = 2$ for every i. But by the same argument as Claim 3.2, we see that this case does not occur.

(ii.3.1.2) If $\det(\mathcal{E})F_5 = 1$, then the number of multiple fibers is at most six.

(ii.3.1.2.1) If the number of its multiple fibers is six, then $m_6 = 2$ and the equality in (14) holds. Hence $m_3 = 2$, $m_4 = 2$, $\det(\mathcal{E})F_1 = \cdots = \det(\mathcal{E})F_4$. Therefore $m_i = 2$ for every integer i with $1 \leq i \leq 6$.

(ii.3.1.2.2) If the number of its multiple fibers is five, then $\det(\mathcal{E})F = \det(\mathcal{E})(2F_5) = 2$, $\det(\mathcal{E})F_1 = \cdots = \det(\mathcal{E})F_4 = 1$ and $m_1 = \cdots = m_4 = 2$. On the other hand

$$\begin{aligned} K_S \det(\mathcal{E}) &= \left(-F_1 - F_2 + \sum_{i \geq 3} (m_i - 1)F_i \right) \det(\mathcal{E}) \\ &= ((m_3 - 2)F_3 + (m_4 - 2)F_4 + F_5) \det(\mathcal{E}) \\ &= 1. \end{aligned}$$

But this is a contradiction.

Therefore if $\det(\mathcal{E})F_5 = 1$, then the number of its multiple fibers is six and $(m_1, \ldots, m_6) = (2, \ldots, 2)$. This is the case (2) in Table 1.

(ii.3.2) The case where $K_S(\det(\mathcal{E})) = 1$.
Then the number of its multiple fibers is five and $m_5 = 2$ and $\det(\mathcal{E})F_5 = 1$. So we have $\det(\mathcal{E})F_1 = \cdots = \det(\mathcal{E})F_5 = 1$ and $m_1 = \cdots = m_5 = 2$. This is the case (3) in Table 1.

(c.1.b.2.2.2) Next we consider the case where $q(S) = g(C)$. Then since $\chi(\mathcal{O}_S) = 0$ by assumption (see (c.1.b)) and $q(S) \leq 2$ (see (c.1.b.2)), we have $1 \leq q(S) \leq 2$.

(i) The case where $q(S) = 2$ and $g(C) = 2$.
Then $K_S \det(\mathcal{E}) = (2F + \sum (m_i - 1)F_i) \det(\mathcal{E})$. Since $K_S \det(\mathcal{E}) \leq 2$, we have $K_S \det(\mathcal{E}) = 2$, $F \det(\mathcal{E}) = 1$ and f has no multiple fiber. In this case $(\det(\mathcal{E}))^2 = 2$. This is the case (32) in Table 1.

(ii) The case where $q(S) = 1$ and $g(C) = 1$.
Then $K_S \det(\mathcal{E}) = \left(\sum_i (m_i - 1)F_i \right) \det(\mathcal{E})$.

(ii.1) If $K_S \det(\mathcal{E}) = 1$, then f has one multiple fiber $2F_1$ and $\det(\mathcal{E})F_1 = 1$. This is the case (33) in Table 1.

(ii.2) If $K_S \det(\mathcal{E}) = 2$, then one of the following types can occur.

(ii.2.1) f has two multiple fibers $2F_1$, $2F_2$ and $\det(\mathcal{E})F_1 = \det(\mathcal{E})F_2 = 1$. This is the case (34) in Table 1.

(ii.2.2) f has one multiple fiber $2F_1$ and $\det(\mathcal{E})F_1 = 2$. This is the case (35) in Table 1.

(ii.2.3) f has one multiple fiber $3F_1$ and $\det(\mathcal{E})F_1 = 1$. This is the case (36) in Table 1.

(c.2) Assume that $\kappa(S) = -\infty$.
Then $h^0(K_X + (n-1)L) = g(S, \det(\mathcal{E})) - q(S)$. Hence

$$g(S,\det(\mathcal{E}))=q(S)+2. \tag{15}$$

Here we note that $(S,\det(\mathcal{E}))$ is not a scroll over a smooth curve because $\det(\mathcal{E})B\geq 2$ for every rational curve B on S. Therefore by [10, Lemma 1.16] we have $g(S,\det(\mathcal{E}))\geq 2q(S)$ and $q(S)\leq 2$. On the other hand since $K_S+\det\mathcal{E}$ is nef, we have

$$\begin{aligned} 0 &\leq (K_S+\det(\mathcal{E}))^2 \\ &= K_S^2+4g(S,\det(\mathcal{E}))-4-(\det(\mathcal{E}))^2. \end{aligned} \tag{16}$$

If $q(S)>0$, then $K_S^2\leq 8(1-q(S))$. Hence we have

$$\begin{aligned} 0 &\leq 8(1-q(S))+4(g(S,\det(\mathcal{E}))-1)-(\det(\mathcal{E}))^2 \\ &= 4(g(S,\det(\mathcal{E}))-2q(S))+4-(\det(\mathcal{E}))^2. \end{aligned} \tag{17}$$

Here we will divide three cases.

(c.2.1) The case where $q(S)=2$.
Since we see from (15) that $g(S,\det(\mathcal{E}))=2q(S)$ in this case, we have $(\det(\mathcal{E}))^2\leq 4$ from (17). We note that there exists a fiber space $h:S\to C$ over a smooth projective curve C with $g(C)=q(S)=2$ such that any general fiber of h is $\mathbb{P}^1$ since $q(S)>0$.

(c.2.1.1) We consider the case where S is minimal. Then there exists a vector bundle $\mathcal{F}$ of rank two on C such that $\mathcal{F}$ is normalized and $S\cong\mathbb{P}_C(\mathcal{F})$. Let C_0 be the minimal section of $h:S\to C$. We set $f:=-\deg\mathcal{F}$. Then $C_0^2=-f$ and $K_S\equiv -2C_0+(2-f)F$. We can write $\det(\mathcal{E})\equiv aC_0+bF$, where F is a general fiber of h. Since $\det(\mathcal{E})$ is ample, we get the following (see [18, Corollary 2.18 and Proposition 2.21 in Chapter V]).

(a) If $f\geq 0$, then $a>0$ and $b>af$.

(b) If $f<0$, then $a>0$ and $b>(1/2)af$.

We also note that $(\det(\mathcal{E}))^2=a(2b-af)$. Since $a=\det(\mathcal{E})F\geq 2$, we have $(\det(\mathcal{E}))^2=2,3,4$.

(c.2.1.1.1) Assume that $(\det(\mathcal{E}))^2=2$. Then since $a\geq 2$ and $2b-af\geq 1$, we have $a=2$ and $2b-af=1$. Therefore $1=2b-af=2(b-f)$ but this is impossible.

(c.2.1.1.2) Assume that $(\det(\mathcal{E}))^2=4$. Then $a=2$ or 4. If $a=4$, then $2b-af=1$. But then $1=2b-af=2b-4f$ and this is also impossible. Therefore $a=2$ and $b-f=1$.

(c.2.1.1.3) Assume that $(\det(\mathcal{E}))^2=3$. Then we have $a=3$ and $2b-af=1$. In particular $2b-3f=1$. If $f\geq 0$, then from (a) above, we have $b>af=3f$ and $6f-3f<2b-3f=1$, that is, $f\leq 0$. Therefore $f=0$. On the other hand, we see that $b\notin\mathbb{Z}$ because $2b-3f=1$. Therefore we have $f<0$. Since $(\det(\mathcal{E}))^2=3$ and S is minimal, we have $(K_S+\det(\mathcal{E}))^2=1$. We note that $K_S+\det(\mathcal{E})\equiv(a-2)C_0+(2+b-f)F$. So we have $(K_S+\det(\mathcal{E}))^2=-(a-2)^2f+2(a-2)(2+b-f)$. Since $a=3$ and $2b-3f=1$, we have $(K_S+\det(\mathcal{E}))^2=2b-3f+4=5$ and this is also impossible.

Therefore $(\det(\mathcal{E}))^2=4$, $a=2$ and $b-f=1$. Then the rank of $\mathcal{E}$ is two because $\det(\mathcal{E})F=2$. Hence $n=3$. In this case $L^3+c_2(\mathcal{E})=(\det(\mathcal{E}))^2=4$. Here we note that $h^*h_*(\mathcal{E}\otimes H(\mathcal{F})^{-1})\to\mathcal{E}\otimes H(\mathcal{F})^{-1}$ is surjective. Since $h^0((\mathcal{E}\otimes H(\mathcal{F})^{-1})|_{F_h})=h^0(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1})=2$ for any fiber F_h of h, we see that $h_*(\mathcal{E}\otimes H(\mathcal{F})^{-1})$ is a locally free sheaf of rank two. Therefore the above homomorphism is an isomorphism. Let $\mathcal{G}:=h_*(\mathcal{E}\otimes H(\mathcal{F})^{-1})$. Then $\mathcal{E}\cong h^*(\mathcal{G})\otimes H(\mathcal{F})$. We note that $\deg\mathcal{G}=f+1$ because $\det(\mathcal{E})=2H(\mathcal{F})+(\deg\mathcal{G})F$. Since $c_2(\mathcal{E})=c_2(h^*(\mathcal{G})\otimes H(\mathcal{F}))=\deg\mathcal{G}-f=1$, we have $L^3=c_1(\mathcal{E})^2-c_2(\mathcal{E})=3$.

Since $\det \mathcal{E}$ is ample, we get the following: If $f < 0$, then we have $f \geq -g(C)$ (see [27]), that is, $f = -1, -2$. If $f \geq 0$, then by (a) above we have $f + 1 > 2f$. Namely we have $f = 0$. Therefore $f = 0, -1, -2$ and $\deg \mathcal{G} = 1, 0, -1$ respectively.

Next we prove the following.

Claim 3.3 *If $f \leq -1$, then $\mathcal{F}$ and $\mathcal{G}$ are semistable.*

Proof. Assume that $\mathcal{G}$ is not semistable. Then there exists a quotient line bundle $\mathcal{Q}$ of $\mathcal{G}$ such that $\mu(\mathcal{G}) > \mu(\mathcal{Q})$, where $\mu(\mathcal{G}) = \deg(\mathcal{G})/\mathrm{rank}(\mathcal{G})$ and $\mu(\mathcal{Q}) = \deg(\mathcal{Q})/\mathrm{rank}(\mathcal{Q})$. Here we note that $\mu(\mathcal{Q}) = c_1(\mathcal{Q})$ and $\mu(\mathcal{G}) = (1/2)c_1(\mathcal{G})$. Hence $0 < \mu(\mathcal{G}) - \mu(\mathcal{Q}) = (1/2)c_1(\mathcal{G}) - c_1(\mathcal{Q})$. Namely $2c_1(\mathcal{Q}) < c_1(\mathcal{G}) = f + 1 \leq 0$. Then since $\mathcal{G} \to \mathcal{Q} \to 0$ is exact, $h^*(\mathcal{G}) \otimes H(\mathcal{F}) \to h^*(\mathcal{Q}) \otimes H(\mathcal{F}) \to 0$ is also exact. Since $\mathcal{E} = h^*(\mathcal{G}) \otimes H(\mathcal{F})$ is ample, so is $h^*(\mathcal{Q}) \otimes H(\mathcal{F})$. Therefore $0 < c_1(h^*(\mathcal{Q}) \otimes H(\mathcal{F}))^2 = (H(\mathcal{F}) + c_1(\mathcal{Q})F)^2 = -f + 2c_1(\mathcal{Q}) \leq 0$. But this is impossible. Therefore $\mathcal{G}$ is semistable.

Next we consider the semistability of $\mathcal{F}$. Let $\mathcal{Q}'$ be a quotient line bundle of $\mathcal{F}$ and let Z be the section of h corresponding to $\mathcal{Q}'$. Then $0 < c_1(\mathcal{E})Z = (2H(\mathcal{F}) + (f+1)F)Z = 2c_1(\mathcal{Q}') + f + 1 \leq 2c_1(\mathcal{Q}')$. Namely $c_1(\mathcal{Q}') > 0$. Since $\deg \mathcal{F} = -f$, we see that $\mu(\mathcal{Q}') - \mu(\mathcal{F}) = c_1(\mathcal{Q}') - (1/2)c_1(\mathcal{F}) \geq 1 + \frac{1}{2}f \geq 0$ because $f = -1$ or -2. Therefore $\mathcal{F}$ is semistable. This completes the proof. $\square$

This is the case (B.6) in Theorem 3.1.

(c.2.1.2) Next we consider the case where S is not minimal.
Then from (17) we have $1 \leq (\det(\mathcal{E}))^2 \leq 3$.

Assume that $(\det(\mathcal{E}))^2 = 1$ (resp. 2, 3). Then we see from (16) that $-11 \leq K_S^2 \leq -9$ (resp. $-10 \leq K_S^2 \leq -9$, $K_S^2 = -9$). Since $g(S, \det(\mathcal{E})) = 4$, we have $K_S \det(\mathcal{E}) = 5$ (resp. 4, 3). Hence $(2K_S + \det(\mathcal{E}))^2 = 4K_S^2 + 4K_S \det(\mathcal{E}) + (\det(\mathcal{E}))^2 < 0$, that is, $2K_S + \det(\mathcal{E})$ is not nef. Therefore there exists an extremal rational curve E on S such that $(2K_S + \det(\mathcal{E}))E < 0$. Since $q(S) = 2$, we see that E is a (-1)-curve. Hence $K_S E = -1$ and $\det(\mathcal{E})E = 1$. But this is impossible.

(c.2.2) The case where $q(S) = 1$.
Since $g(S, \det(\mathcal{E})) = q(S) + 2 = 3$, by [21, (2.3) Theorem (V)] we get the nine types. This is the case (B.7) in Theorem 3.1.

(c.2.3) The case where $q(S) = 0$.
Since $g(S, \det(\mathcal{E})) = q(S) + 2 = 2$, by [7, (2.25) Theorem] we get the three types 4), 5_0) and 5_1). For detail, see [7, (2.25) Theorem]. This is the case (B.8) in Theorem 3.1.

(d) If (M, A) is $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ (resp. $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$, $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$), then by (d.1) (resp. (d.2), (d.3)) in the proof of [14, Theorem 2.2] we have $h^0(K_X + 3L) = 5$ (resp. $h^0(K_X + 2L) = 5$, $h^0(K_X + 2L) = 10$). So these cases cannot occur. Next we consider the case in which (M, A) is a $\mathbb{P}^2$-bundle over a smooth curve with $L_F \cong \mathcal{O}_{\mathbb{P}^2}(2)$ for any fiber F.
Here we use notation in [14, Theorem 2.2]. By (d.4) in the proof of [14, Theorem 2.2], we have $g(C) = 1$. Then $h^0(K_X + 2L) = e$, that is $e = 2$. Moreover $g(X, L) = 1 + e + 2 - 2g(C) = 3$. Hence by [22, (3.4) Theorem (II)], we see that $X \cong \mathbb{P}_C(\mathcal{E})$, $\mathcal{E} = \pi_* \mathcal{O}(K_X + 2L)$ is a stable vector bundle of rank three with $c_1(\mathcal{E}) = 2$ and $L = 2H(\mathcal{E}) + \pi^* B$, where $B \in \mathrm{Pic}(C)$ with $\det(\mathcal{E}) + 2B = 0$, where $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ is the natural map. This is the case (C) in Theorem 3.1.

Therefore we get the assertion. $\square$

Remark 3.1 Here we consider an example of some types in Theorem 3.1.
(a) There exists an example of the type (A.2) if $d = 3, 4$ or 6. See [22, (2.4)].

(b) There exists an example of each case in (A.3). See [6, from (3.15) to (3.29)].

(c) An example of the type (B.2) in Theorem 3.1 (see also [11, Example 4.7]).
First let E be a smooth elliptic curve. Then there exists an indecomposable and ample vector bundle $\mathcal{G}$ of rank two on E such that $c_1(\mathcal{G}) = 1$. Let $S := E \times E'$, where E' is an arbitrary smooth elliptic curve. We set $\mathcal{E} := p_1^*(\mathcal{G}) \otimes p_2^*(D)$, where p_i is the natural projection and D is a line bundle with $\deg D = 1$. Then this $(S, \mathcal{E})$ is an example of the type (B.2) in Theorem 3.1.

(d) An example of the type (B.3) in Theorem 3.1.
First we note that there exist smooth elliptic curves A and B, and an abelian group G such that $S \cong (A \times B)/G$. Then the types of G and the basis of $\mathrm{Num}(S)$ are the following (see [28, Tables 1 and 2]).

G	$\mathrm{Num}(S)$
$\mathbb{Z}_2$	$(1/2)A$, B
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(1/2)A$, $(1/2)B$
$\mathbb{Z}_4$	$(1/4)A$, B
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$(1/4)A$, $(1/2)B$
$\mathbb{Z}_3$	$(1/3)A$, B
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$(1/3)A$, $(1/3)B$
$\mathbb{Z}_6$	$(1/6)A$, B

We also note that $AB = \gamma$, where $\gamma := |G|$. First we claim the following.

Claim 3.4 *$\mathcal{E}$ is indecomposable.*

Proof. Assume that $\mathcal{E}$ is decomposable. Then $c_1(\det \mathcal{E}) = L_1 + L_2$ for two ample line bundles L_1 and L_2 on S. Let αA and βB be the basis of $\mathrm{Num}(S)$, where $\alpha, \beta \in \mathbb{Q}$. Then the above table shows that $\alpha\beta\gamma = 1$. Let $L_i := x_i \alpha A + y_i \beta B$ for $i = 1, 2$. Then $x_i > 0$ and $y_i > 0$ by [28, Lemma 1.3]. Hence $L_i^2 = 2x_i y_i \geq 2$ and $c_1(\mathcal{E})^2 = L_1^2 + L_2^2 + 2L_1L_2 > 4$. But this is a contradiction. Hence $\mathcal{E}$ is indecomposable. □

Here let $G = \mathbb{Z}_2$. Set $S := (A \times B)/G$, where G acts on $A \times B$ componentwise. Then there exist two fiber spaces $\Phi : S \to A/G$ and $\Psi : S \to B/G$ such that A/G is a smooth elliptic curve and B/G is a smooth rational curve. Here we note that any smooth fiber of Φ (resp. Ψ) is isomorphic to B (resp. A). We also note that $AB = 2$. Then we can take an indecomposable ample vector bundle $\mathcal{F}$ of rank two on A/G such that $c_1(\mathcal{F}) = 1$. Moreover Ψ has a multiple fiber $2F'$ whose multiplicity is two (see [28, Table 2 in Theorem 1.4]). Then we set $\mathcal{E} := \Phi^*(\mathcal{F}) \otimes \mathcal{O}(F')$. Then we prove the following.

Claim 3.5 *$\mathcal{E}$ is ample.*

Proof. Let $\pi_1 : \mathbb{P}_{A/G}(\mathcal{F}) \to A/G$. Then there exists a morphism $q_1 : \mathbb{P}_S(\Phi^*(\mathcal{F})) \to \mathbb{P}_{A/G}(\mathcal{F})$ such that $\Phi \circ \pi = \pi_1 \circ q_1$, where $\pi : \mathbb{P}_S(\Phi^*(\mathcal{F})) \to S$. Then $q_1^* H(\mathcal{F}) = H(\Phi^*(\mathcal{F}))$. On the other hand $i : \mathbb{P}_S(\mathcal{E}) \to \mathbb{P}_S(\Phi^*(\mathcal{F}))$ is an isomorphism and $i_*(H(\mathcal{E})) \otimes \pi^*\mathcal{O}(-F') = H(\Phi^*(\mathcal{F}))$ (see [18, Lemma 7.9 in Chapter II]). Therefore $\mathcal{E}$ is ample if and only if $H(\Phi^*(\mathcal{F})) \otimes \pi^*\mathcal{O}(F')$ is ample.

Here we note that $2q_1^*H(\mathcal{F}) \otimes \pi^*\mathcal{O}(2F') = 2q_1^*H(\mathcal{F}) \otimes r^*\mathcal{O}(P)$ for some point $P \in B/G$, where we set $r := \Psi \circ \pi : \mathbb{P}_S(\Phi^*(\mathcal{F})) \to S \to B/G$. Let $H := 2q_1^*H(\mathcal{F}) \otimes r^*\mathcal{O}(P)$. Then it suffices to show that H is ample because $H(\Phi^*(\mathcal{F})) \otimes \pi^*\mathcal{O}(F') = q_1^*H(\mathcal{F}) \otimes \pi^*\mathcal{O}(F')$. Here we use [5, Theorem B6 in Appendix B]. Let Y be an irreducible subvariety of $\mathbb{P}_S(\Phi^*(\mathcal{F}))$. Here we note that $\dim \mathbb{P}_S(\Phi^*(\mathcal{F})) = 3$.

(1) Assume that $\dim Y = 3$. Since $H(\mathcal{F})$ and $\mathcal{O}(P)$ are ample, we see that $h^0(mH) > 0$ for $m \gg 0$ and $|mH|$ has a nonzero effective divisor.

(2) Assume that $\dim Y = 2$.

(2.1) If $r(Y)$ is not a point, then $r(Y) = B/G$. For some $m \gg 0$, we have $\mathrm{Bs}|2mH(\mathcal{F})| = \emptyset$ and $\mathrm{Bs}|\mathcal{O}(mP)| = \emptyset$. Hence by Bertini's theorem, there exist $D_1 \in |q_1^*(2mH(\mathcal{F}))|$ and $D_2 \in |r^*(\mathcal{O}(mP))|$ such that $D_1 \not\supset Y$ and $D_2 \not\supset Y$. Since $r(Y) = B/G$, we have $D_2 \cap Y \neq \emptyset$. Therefore $D_1 + D_2 \in |(mH)|_Y|$ is a nonzero effective divisor.

(2.2) If $r(Y)$ is a point, then Y is contained in a fiber of r. Let $F = \pi(Y)$. Then F and $\Phi(F)$ are curves. Namely $\Phi(F) = A/G$. Since $\Phi \circ \pi = \pi_1 \circ q_1$ and $\Phi \circ \pi(Y)$ is not a point, we see that $q_1(Y)$ is not a point. Hence by the same argument as in the case (2.1) we see that for $m \gg 0$ there exist $D_1 \in |q_1^*(2mH(\mathcal{F}))|$ and $D_2 \in |r^*(\mathcal{O}(mP))|$ such that $D_1 \not\supset Y$ and $D_2 \not\supset Y$. Moreover $D_1 \cap Y \neq \emptyset$ because D_1 is ample. Therefore $D_1 + D_2 \in |mH|_Y|$ is a nonzero effective divisor.

(3) Assume that $\dim Y = 1$.

(3.1) If Y is not contained in any fiber of r, then by the same method as in the case (2.1) above, $|mH|_Y|$ has a nonzero effective divisor for $m \gg 0$.

(3.2) Assume that Y is contained in a fiber of r.

(3.2.1) If $\pi(Y)$ is a curve, then by the same argument as in the case (2.1), we see that $|mH|_Y|$ has a nonzero effective divisor for $m \gg 0$.

(3.2.2) If $\pi(Y)$ is a point, then Y is a fiber of π. We also note that $q_1(Y)$ is a curve by construction. Here we take a positive integer m such that $\mathrm{Bs}|2mH(\mathcal{F})| = \emptyset$ and $\mathrm{Bs}|\mathcal{O}(mP)| = \emptyset$. Then there exist $D_1 \in |q_1^*(2mH(\mathcal{F}))|$ and $D_2 \in |r^*(\mathcal{O}(mP))|$ such that $D_1 \not\supset Y$ and $D_2 \not\supset Y$. Moreover $D_1 \cap Y \neq \emptyset$. Therefore $D_1 + D_2 \in |mH|_Y|$ is a nonzero effective divisor.

Hence we see from the above argument that mH is ample, that is, H is ample. This implies that $\mathcal{E}$ is also ample. □

We note that $c_1(\mathcal{E}) = \Phi^* c_1(\mathcal{F}) + 2F'$ and $c_1(\mathcal{E})^2 = 4$. Therefore this $(S, \mathcal{E})$ is an example of (B.3) in Theorem 3.1.

(e) We consider the case (B.5) in Theorem 3.1.

(e.1) First we consider the case where $q(X) = 2$ and $g(C) = 1$. Let $\alpha : S \to \mathrm{Alb}(S)$ be the Albanese map of S. Since $q(S) = 2$, we have $\dim \mathrm{Alb}(S) = 2$. Assume that $\dim \alpha(S) = 1$. Then $\alpha(S)$ is a smooth curve of genus 2 and $\alpha : S \to \alpha(S)$ is a fiber space. Since $\kappa(S) = 1$, we see that any general fiber of α is an elliptic curve. Here we note that for any general fiber F of $f : S \to C$, $\alpha(F)$ is a point because $g(\alpha(S)) = 2$. Hence by [1, Lemma 4.1.13], there exists a morphism $\delta : C \to \alpha(S)$ such that $\alpha = \delta \circ f$. But this is impossible because $g(C) < g(\alpha(S))$. Hence $\dim \alpha(S) = 2$. Namely α is surjective. For any general fiber F of f, $\alpha(F)$ is a curve. We also note that $\alpha(F)$ is a complex subtorus of $\mathrm{Alb}(S)$ (see [31, Theorem 10.3]). Let $H := \alpha(F)$ for a fixed general fiber F of f. We consider $\pi : \mathrm{Alb}(S) \to \mathrm{Alb}(S)/H$. Here we note that $\mathrm{Alb}(S)/H$ is an Abelian variety of dimension 1, that is, a smooth elliptic curve (see also [23, (4.2) Proposition in Chapter 2 and (1.1) Proposition in Chapter 4]). Then since $FF' = 0$ for any general fiber F' of f and $(\pi \circ \alpha)(F)$ is a point, we infer that $(\pi \circ \alpha)(F')$ is a point. Hence there exists a morphism $\iota : C \to \mathrm{Alb}(S)/H$ such that $\pi \circ \alpha = \iota \circ f$. Since any fiber of f is irreducible in this case (see [29, Lemmas 1.5 and 1.6]), we see that α is finite. Moreover since mulitplicity of every multiple fiber of f is 2, we also see that $\deg \alpha = 2$. Therefore S is a double cover of an Abelian surface.

(e.2) Next we consider the case where $q(S) = 1$ and $g(C) = 0$. Then this case corresponds to one of the cases from (2) to (31) in Table 1. We use Proposition 2.1 and the notation in this proposition. Then by Proposition 2.1 the generators of $\mathrm{Num}(S)$ are $N_1 = (1/2)F$ and $N_2 = (2/\gamma)D + (\delta/4)F$. Let $\det \mathcal{E} = \alpha N_1 + \beta N_2$ and let $K_S \equiv \epsilon F$. Then by (5) and (6) in Claim 3.2 we have

$$\begin{aligned}(\det(\mathcal{E}))^2 &= \beta(2\alpha + \beta\delta), \\ K_S(\det(\mathcal{E})) &= \epsilon\mu\beta,\end{aligned}$$

First we consider the case of (2) in Table 1.
In this case $\mu = 2$. Since $K_S \equiv F$, we have $2 = K_S(\det \mathcal{E}) = 2\beta$ because $DF = \gamma$. Hence $\beta = 1$. We also note that

$$2 = (\det \mathcal{E})^2 = (2\alpha + \beta\delta)\beta = 2\alpha + \delta.$$

Since $\delta = 0$ or 1 by definition, we see that if $\delta = 1$, then this is impossible because $2 = 2\alpha + \delta$. Hence $\delta = 0$ and $\alpha = 1$.

By the same argument as above, we can get $(\mu, \epsilon, \alpha, \beta, \delta)$ for other cases. See the Table 2 below.

(f) We consider the case (B.6) in Theorem 3.1.

(f.1) An example of the case where $\deg(\mathcal{F}) = 0$.
First let $\mathcal{F}$ be a normalized vector bundle on C with $\deg \mathcal{F} = 0$. Then $H(\mathcal{F})$ is nef. Let $\mathcal{G}$ be a stable vector bundle of rank two on C with $\deg \mathcal{G} = 1$. This vector bundle always exists (see e.g. [26, Theorem 8.6.1]). Then we note that $\mathcal{G}$ is ample (see [25, Main Claim in p.62]). Let $\mathcal{E} := h^*(\mathcal{G}) \otimes H(\mathcal{F})$. Then $\mathcal{E}$ is ample by the method similar to [7, (2.6)] and this is an example of the case where $\deg(\mathcal{F}) = 0$.

(f.2) An example of the case where $\deg(\mathcal{F}) = 1$.
First let $\mathcal{F}$ be a normalized vector bundle on C with $\deg \mathcal{F} = 1$. Then $H(\mathcal{F})$ is ample. Let $\mathcal{E} := H(\mathcal{F}) \oplus H(\mathcal{F})$ and this is an example of the case where $\deg(\mathcal{F}) = 1$.

(f.3) Finally we consider the case where $\deg(\mathcal{F}) = 2$.
Let $\mathcal{F}$ and $\mathcal{G}$ be semistable vector bundles on C such that $\deg \mathcal{F} = 2$ and $\deg \mathcal{G} = -1$. (Here we note that these vector bundles always exist. See e.g. [26, Theorem 8.6.1].) Let $\mathcal{E} := h^*(\mathcal{G}) \otimes H(\mathcal{F})$. Then $\mathcal{E}$ is ample by the method similar to [7, (2.6)].

(g) There exists an example of each case in (B.8). See [7].

(h) There exists an example of each case in (C). See [22, (3.3)].

Finally we note that, as we said at the beginning of this section, we can get a classification of polarized 3-folds (X, L) with $h^0(K_X + 2L) = 2$ from Theorem 3.1.

Corollary 3.1 *Let (X, L) be a polarized manifold of dimension 3. If $h^0(K_X + 2L) = 2$, then (X, L) is one of the types in Theorem 3.1.*

4 Final remarks

Let (X, L) be a polarized manifold of dimension $n \geq 3$. In [12] and [13], for every integer i with $0 \leq i \leq n$, we introduced new invariants of (X, L), the *ith sectional geometric genus* $g_i(X, L)$ and the *ith sectional H-arithmetic genus* $\chi_i^H(X, L)$, which play important roles in the study of the

Table 2: The case where $q(S) = 1$ and $g(C) = 0$.

The case in Table 1	μ	ϵ	α	β	δ
(2)	2	1	1	1	0
(3)	2	1/2	1	1	1
(4)	3	2/3	1	1	0
(5)	6	1/3	1	1	0
(6)	4	1/2	1	1	0
(7)	6	1/3	1	1	0
(8)	4	1/4	1	1	1
(9)	6	1/6	1	1	1
(10)	5	2/5	1	1	0
(11)	4	1/4	1	1	1
(12)	6	1/3	1	1	0
(13)	6	1/6	1	1	0
(14)	12	1/6	1	1	0
(15)	12	1/12	1	1	1
(16)	9	2/9	1	1	0
(17)	15	2/15	1	1	0
(18)	30	1/15	1	1	0
(19)	18	1/9	1	1	0
(20)	12	1/6	1	1	0
(21)	8	1/4	1	1	0
(22)	10	1/5	1	1	0
(23)	42	1/42	1	1	1
(24)	24	1/24	1	1	1
(25)	18	1/18	1	1	1
(26)	12	1/12	1	1	1
(27)	20	1/20	1	1	1
(28)	12	1/12	1	1	1
(29)	8	1/8	1	1	1
(30)	6	1/6	1	1	1
(31)	10	1/10	1	1	1

dimension of global sections of adjoint bundle (see [14] and [15]). In particular, the author proved Conjecture 1.1 for $\dim X = 3$ by using the theory of these invariants.

In [12, Conjecture 4.1] and [13, Conjecture 2.1], we proposed the following conjecture.

Conjecture 4.1 *Let (X, L) be a polarized manifold of dimension n. Then the following hold.*

(1) *$g_i(X, L) \geq h^i(\mathcal{O}_X)$ for $n \geq 2$ and $0 \leq i \leq n$.*

(2) *$\chi_2^H(X, L) > 0$ holds if $n \geq 3$ and $\kappa(K_X + (n-2)L) \geq 0$.*

Remark 4.1 We note that by definition

$$\chi_2^H(X, L) = 1 - h^1(\mathcal{O}_X) + g_2(X, L)$$

holds. Hence Conjecture 4.1 (2) implies that $g_2(X, L) \geq h^1(\mathcal{O}_X)$ if $n \geq 3$ and $\kappa(K_X+(n-2)L) \geq 0$.

In [14, Theorem 3.1.1], we proved the following which shows a relation between Conjecture 1.1 and Conjecture 4.1.

Theorem 4.1 *If $n \geq 4$ and Conjecture* 4.1 *is true, then Conjecture* 1.1 *is true.*

Concerning this result, we can also prove the following.

Theorem 4.2 *Let (X, L) be a polarized manifold of dimension n. Assume that $n \geq 4$ and Conjecture* 4.1 *is true. If $h^0(K_X + (n-1)L) \leq 2$, then $\kappa(K_X + (n-2)L) = -\infty$.*

Proof. We use an invariant $A_i(X, L)$ which was defined in [16, Definition 3.2]. This invariant has the following properties.

(A) (See [16, Proposition 3.2 and Remark 3.2].) For every integer i with $1 \leq i \leq n$, we have $A_i(X, L) = g_i(X, L) + g_{i-1}(X, L) - h^{i-1}(\mathcal{O}_X)$ and $A_0(X, L) = L^n$. In particular $A_n(X, L) = h^0(K_X + L)$.

(B) (See [16, Corollary 3.1].) For every positive integer t, we have

$$h^0(K_X + tL) = \sum_{j=0}^{n} \binom{t-1}{n-j} A_j(X, L).$$

From (B) above, we have

$$h^0(K_X + (n-1)L) = \sum_{j=2}^{n} \binom{n-2}{n-j} A_j(X, L).$$

Assume that $\kappa(K_X + (n-2)L) \geq 0$. Then from (A) above and the assumption that Conjecture 4.1 holds, we see that $A_j(X, L) \geq 0$ for every integer j with $3 \leq j \leq n$ and $A_2(X, L) \geq g_1(X, L)$. Here we note that $g_1(X, L)$ is the sectional genus of (X, L) (see [12, Remark 2.1.1 (1)]). Hence $h^0(K_X + (n-1)L) \geq g_1(X, L)$. So we get $g_1(X, L) \leq 2$ by assumption. On the other hand, since $\kappa(K_X + (n-2)L) \geq 0$, we have $g_1(X, L) = 1 + (1/2)(K_X + (n-1)L)L^{n-1} \geq 2$ because $L^n > 0$ and $g_1(X, L)$ is an integer. Therefore $g_1(X, L) = 2$. By [6, (1.10) Theorem] and [14, Theorem 1.6 and Remark 1.7], we see that (X, L) is one of the types (I), (II) and (III) in [14, Theorem 1.6]. If (X, L) is the type (I) in [14, Theorem 1.6], then

$$g_i(X, L) = \begin{cases} 0 & \text{for every integer } i \text{ with } 4 \leq i \leq n, \\ 1 & \text{if } i = 3, \end{cases} \tag{18}$$

$$h^j(\mathcal{O}_X) = 0 \quad \text{for every integer } j \text{ with } 1 \leq j \leq n. \tag{19}$$

Moreover by Conjecture 4.1 we have $g_2(X,L)\geq 0$. Therefore

$$\begin{aligned}
A_2(X,L) &= g_2(X,L)+g_1(X,L)-h^1(\mathcal{O}_X)\geq 2 && (20)\\
A_3(X,L) &= g_3(X,L)+g_2(X,L)-h^2(\mathcal{O}_X)=g_2(X,L)+1\geq 1 && (21)\\
A_4(X,L) &= g_4(X,L)+g_3(X,L)-h^3(\mathcal{O}_X)=1 && (22)\\
A_j(X,L) &= 0 \text{ for every integer } j \text{ with } 4\leq j\leq n. && (23)
\end{aligned}$$

Hence by (20), (21), (22) and (23) we have

$$\begin{aligned}
h^0(K_X+(n-1)L) &= \sum_{j=0}^{n}\binom{n-2}{n-j}A_j(X,L)\\
&= A_2(X,L)+(n-2)A_3(X,L)+\binom{n-2}{2}A_4(X,L)\\
&\geq 2+\binom{n-1}{2}\geq 5.
\end{aligned}$$

If (X,L) is of the type (II) in [14, Theorem 1.6], then $K_X+(n-1)L=\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))$. (Here we use the notation in [14, Theorem 1.6].) Hence

$$\begin{aligned}
h^0(K_X+(n-1)L) &= h^0(\pi^*(\mathcal{O}_{\mathbb{P}^n}(1)))\\
&= h^0(\mathcal{O}_{\mathbb{P}^n}(1)\oplus\mathcal{O}_{\mathbb{P}^n}(-2))\\
&= h^0(\mathcal{O}_{\mathbb{P}^n}(1))\\
&= n+1\geq 5.
\end{aligned}$$

Hence this contradicts the assumption.

If (X,L) is of the type (III) in [14, Theorem 1.6], then (X,L) is a simple blowing up of a polarized manifold (X',L') which is of the type (II) in [14, Theorem 1.6]. Since $h^0(K_X+(n-1)L)=h^0(K_{X'}+(n-1)L')$, by the argument above we get

$$\begin{aligned}
h^0(K_X+(n-1)L) &= h^0(K_{X'}+(n-1)L')\\
&= n+1\geq 5.
\end{aligned}$$

Hence this also contradicts the assumption. Therefore we get the assertion. □

Remark 4.2 In [6, Remark (2.2)] Fujita conjectured that $h^0(L)>0$ if (X,L) is the type (I) in [14, Theorem 1.6]. Here we note that $h^0(L)>0$ if and only if $g_2(X,L)\geq 3-n$.

Proof. Since

$$h^0(L)=h^0(K_X+(n-2)L)=\sum_{j=0}^{n}\binom{n-3}{n-j}A_j(X,L),$$

by (21), (22) and (23) we have

$$h^0(L)=A_3(X,L)+(n-3)A_4(X,L)=g_2(X,L)+n-2.$$

Therefore we get the assertion. □

In particular, if we can prove that $g_2(X,L)\geq 0$, then this conjecture is true.

By Theorem 4.2 we get the following.

Corollary 4.1 *Let (X,L) be a polarized manifold of dimension $n\geq 4$. Assume that Conjecture* 4.1 *is true. If $h^0(K_X+(n-1)L)=2$, then (X,L) is one of the types in Theorem* 3.1.

5 Appendix

Here we consider the case of $n = 3$. By [14, Theorem 2.3], we see that if $\kappa(K_X + L) \geq 0$, then $h^0(K_X + 2L) \geq 3$. So it is interesting to consider a classification of polarized 3-folds (X, L) with $\kappa(K_X + L) \geq 0$ and $h^0(K_X + 2L) = 3$. In this case, we can prove the following.

Theorem 5.1 *Let (X, L) be a polarized 3-fold. Assume that $\kappa(K_X+L) \geq 0$. Then $h^0(K_X+2L) = 3$ if and only if (X, L) satisfies the following.*

$$(*) \qquad \mathcal{O}_X(K_X) = \mathcal{O}_X, \quad L^3 = 1, \quad h^0(L) = 1 \ \textit{ and } \ q(X) = 0.$$

Proof. (A) We are going to prove the "only if" part. First we note that

$$g_1(X, L) \geq 2 \tag{24}$$

because $\kappa(K_X + L) \geq 0$. We also note that the following inequality holds by [13, Theorem 3.2.1 and Theorem 3.3.1 (2)].

$$g_2(X, L) \geq h^1(\mathcal{O}_X) \tag{25}$$

Furthermore we can get the following.

Claim 5.1 *$h^0(K_X + L) > 0$ holds.*

Proof. First of all, by a result of Höring the following holds.

Theorem 5.2 ([19]) *Let (X, L)be a polarized 3-fold. If K_X+L is nef, then $h^0(K_X+L) > 0$ holds.*

By the assumption that $n = 3$ and $\kappa(K_X + L) \geq 0$, Theorem 2.1 implies that there exist a polarized 3-fold (M, A) and a birational morphism $\mu : X \to M$ such that the following properties hold.

(1) $h^0(K_X + L) = h^0(K_M + A)$.

(2) $K_M + A$ is nef.

Moreover by the condition (2) above and Theorem 5.2, we have $h^0(K_M + A) > 0$. Therefore the condition (1) above implies that $h^0(K_X + L) > 0$ holds. $\square$

Since

$$3 = h^0(K_X + 2L) = h^0(K_X + L) + g_1(X, L) + g_2(X, L) - h^1(\mathcal{O}_X)$$

by [14, Theorem 2.1], we see from (24), (25) and Claim 5.1 that (X, L) satisfies the following.

$$g_1(X, L) = 2, \quad h^0(K_X + L) = 1, \quad g_2(X, L) = h^1(\mathcal{O}_X).$$

By [14, Theorem 1.6 and Remark 1.7] we see that (X, L) is (I), (II) or (III) in [14, Theorem 1.6].

(A.1) If (X, L) is the type (II) or the type (III) in [14, Theorem 1.6], then $h^3(\mathcal{O}_X) = 0$, $h^2(\mathcal{O}_X) = 0$, and $h^0(K_X+L) = 1$. Hence $g_2(X, L) = 1$ by [14, Theorem 1.1 (2)]. On the other hand $h^1(\mathcal{O}_X) = 0$ in these cases. Therefore $g_2(X, L) \neq h^1(\mathcal{O}_X)$ and the types (II) and (III) in [14, Theorem 1.6] do not occur.

(A.2) If (X, L) is the type (I) in [14, Theorem 1.6], then (X, L) satisfies $(*)$ in Theorem 5.1.

(B) Next we are going to prove the "if" part. We assume that (X, L) satisfies $(*)$ in Theorem 5.1.

Then we can easily check that $h^0(K_X+L)=h^0(L)=1$ and $\kappa(K_X+L)\geq 0$. Moreover by [14, Theorem 1.1 (2)] and the Serre duality, we have

$$\begin{aligned} g_2(X,L) &= h^0(K_X+L)-h^3(\mathcal{O}_X)+h^2(\mathcal{O}_X) \\ &= h^0(L)-h^0(\mathcal{O}_X)+h^1(\mathcal{O}_X)=0. \end{aligned}$$

Moreover we have

$$g_1(X,L)=1+\frac{1}{2}(K_X+2L)L^2=1+L^3=2.$$

Therefore

$$\begin{aligned} h^0(K_X+2L) &= h^0(K_X+L)+g_1(X,L)+g_2(X,L)-h^1(\mathcal{O}_X) \\ &= 1+2+0-0=3. \end{aligned}$$

So we get the assertion. □

Example 5.1 Here we note that there exists an example of polarized 3-fold (X,L) with $(*)$ in Theorem 5.1.

Let $\widetilde{X}$ be a quintic hypersurface in $\mathbb{P}^4$ and let $\widetilde{L}=\mathcal{O}_{\widetilde{X}}(1)$. Let $G:=\{1,\xi,\xi^2,\xi^3,\xi^4\}$ be a cyclic subgroup of order 5 of the multiplicative group of $\mathbb{C}$, where ξ is a primitive 5th root of unity. Then we define an action of G on $\mathbb{P}^4$ by

$$g\cdot(x_0:x_1:x_2:x_3:x_4):=(x_0:gx_1:g^2x_2:g^3x_3:g^4x_4)$$

for any $g\in G$. Then G acts freely on $\widetilde{X}$, and $X:=\widetilde{X}/G$ is a smooth projective 3-fold with $\mathcal{O}(K_X)=\mathcal{O}_X$ and $h^j(\mathcal{O}_X)=0$ for $j=1$ and 2. Let $D\in|\widetilde{L}|$. Then $L:=\pi(D)$ is an ample divisor on X with $L^3=1$ and $h^0(L)=1$, where $\pi:\widetilde{X}\to X$ be the quotient map. Hence $g_2(X,L)=h^0(K_X+L)-h^3(\mathcal{O}_X)+h^2(\mathcal{O}_X)=h^0(L)-h^0(\mathcal{O}_X)+h^2(\mathcal{O}_X)=0$ and $g_1(X,L)=1+(1/2)(K_X+2L)L^2=1+L^3=2$. Therefore $h^0(K_X+2L)=h^0(K_X+L)+g_2(X,L)-h^1(\mathcal{O}_X)+g_1(X,L)=3$.

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