偏極多様体の断面種数による分類*†‡

福間　慶明　(高知大学理学部)

1 Introduction

(X, L) を複素数体上定義された n 次元非特異偏極多様体とする．この偏極多様体の古典的不変量として知られているものに, 次数 L^n や断面種数 $g(X, L)$ がある．特にここでは断面種数に関することについて考える．今現在, 断面種数について知られていることとして次があげられる.

(1) $g(X, L)$ は非負整数である ([3], [13]).

(2) $g(X, L)$ の値の小さい場合の (X, L) の分類 ([3], [13], [4]).

また, 藤田隆夫氏により次のことが予想されている.

Conjecture 1.1 ([5]. [1, Question 7.2.11] も参照のこと) *任意の偏極多様体 (X, L) に対して不等式 $g(X, L) \geq h^1(\mathcal{O}_X)$ が成立する.*

L が基点自由のときは, この予想は正しいことが知られている ([1, Theorem 7.2.10] などを参照のこと)．しかし L が一般の豊富な因子の場合には $n = 2$ のときでさえこの予想が正しいかはわかっていない.

そこでもし $g(X, L) \geq h^1(\mathcal{O}_X)$ が成立する場合に $g(X, L) - h^1(\mathcal{O}_X)$ の値による分類がどのようになるかを考えることは興味深い問題である.

まず ***L* が基点自由の場合**には $g(X, L) - h^1(\mathcal{O}_X) = 0, 1, 2$ なる (X, L) の分類が知られている ([6], [7], [8] を参照)．しかし $g(X, L) - h^1(\mathcal{O}_X) \geq 3$ の場合に分類をすることは今のところ技術的に困難である．そこでこの場合には ***L* が非常に豊富であると仮定する**ことで分類がえられるのではないかと考え, 実際 $3 \leq g(X, L) - h^1(\mathcal{O}_X) \leq 5$ の場合に分類ができたのでそれを報告する.

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2 Preliminaries

まず随伴束の理論より次のことが言える.

Definition 2.1 (1) Let X (resp. Y) be an n-dimensional projective manifold, and let L (resp. H) be an ample line bundle on X (resp. Y). Then (X, L) is called a *simple blowing up of* (Y, H) if there exists a birational morphism $\pi : X \to Y$ such that π is a blowing up at a point of Y and $L = \pi^*(H) - E$, where E is the π-exceptional reduced divisor.
(2) Let X (resp. M) be an n-dimensional projective manifold, and let L (resp. A) be an ample line bundle on X (resp. M). Then we say that (M, A) is a *reduction of* (X, L) if (X, L) is obtained by a composite of simple blowing ups of (M, A), and (M, A) is not obtained by a simple blowing up of any polarized manifold. The morphism $\mu : X \to M$ is called the *reduction map*.

Remark 2.1 Let (X, L) be a polarized manifold of dimension n and (M, A) a reduction of (X, L). Then we obtain $g(X, L) = g(M, A)$.

Definition 2.2 Let (X, L) be a polarized manifold of dimension n. We say that (X, L) is a *scroll* (resp. *quadric fibration*, *Del Pezzo fibration*) *over a normal variety* Y *of dimension* m if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $K_X + (n - m + 1)L = f^*A$ (resp. $K_X + (n - m)L = f^*A$, $K_X + (n - m - 1)L = f^*A$) for some ample line bundle A on Y.

Theorem 2.1 *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Then (X, L) is one of the following types.*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(3) *A scroll over a smooth curve.*

(4) $K_X \sim -(n - 1)L$, *that is,* (X, L) *is a Del Pezzo manifold.*

(5) *A quadric fibration over a smooth curve.*

(6) *A scroll over a smooth surface S. In this case, there exists an ample vector bundle $\mathcal{E}$ on S such that $X = \mathbb{P}_S(\mathcal{E})$ and $L = H(\mathcal{E})$.*

(7) *Let (M, A) be a reduction of (X, L).*

(7.1) $n = 4$, $(M, A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.

(7.2) $n = 3$, $(M, A) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.

(7.3) $n = 3$, $(M, A) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.

(7.4) $n = 3$, M *is a* $\mathbb{P}^2$*-bundle over a smooth curve* C *such that* $(F, A|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *for any fiber* F *of it.*

(7.5) $K_M + (n-2)A$ *is nef.*

Proof. See [1, Proposition 7.2.2, Theorem 7.2.4, Theorem 7.3.2, Theorem 7.3.4, and Theorem 7.5.3]. See also [5, Chapter II, (11.2), (11.7), and (11.8)], or [13, Section 1, Theorem]. □

Remark 2.2 Let (X, L) be a polarized manifold of dimension $n \geq 3$. If (X, L) is one of the types from (1) to (6) in Theorem 2.1, then (X, L) is a reduction of itself.

Remark 2.3 Let (X, L) be a polarized manifold of dimension $n \geq 3$. Then $\kappa(K_X + (n-2)L) = -\infty$ if and only if (X, L) is one of the types from (1) to (7.4) in Theorem 2.1.

次の結果が主結果を証明する際に重要となる. Theorem 2.2 は以前私が断面種数の研究を重点的に行っていた際に証明した結果である

Theorem 2.2 *Let* (X, L) *be a polarized manifold with* $\dim X = n \geq 3$. *Assume that* $\dim \mathrm{Bs}|L| \leq 0$, $h^0(L) \geq n + m$, *and* $m \geq 1$, *where* $m = g(X, L) - q(X)$. *Let* (M, A) *be a reduction of* (X, L). *Then* (X, L) *is one of the following types:*

(1) (X, L) *is a Del Pezzo manifold.*

(2) (X, L) *is a hyperquadric fibration over a smooth curve with*

$$q(X) \leq \frac{1}{2} + \frac{2m+1}{2n}.$$

(3) (X, L) *is a scroll over a smooth surface* S *with* $\kappa(S) = -\infty$.
If $q(X) > 0$ *and* S *is relatively minimal, then*

$$q(X) \leq 1 + \frac{2m - n + 1}{n^2 - 3n + 4}.$$

If $q(X) > 0$ *and* S *is not relatively minimal, then*

$$q(X) \leq 1 + \frac{4m - 4n + 3}{8n^2 - 20n + 16}.$$

(4) $(M, A) = (\mathbb{P}^4, \mathcal{O}(2))$.

(5) $(M, A) = (\mathbb{Q}^3, \mathcal{O}(2))$.

(6) $(M, A) = (\mathbb{P}^3, \mathcal{O}(3))$.

(7) *M is a $\mathbb{P}^2$-bundle over a smooth curve C with $(F, A_F) = (\mathbb{P}^2, \mathcal{O}(2))$ for any fiber F of it.*

Proof. [9] を参照のこと. □

さらに上記の Theorem 2.2 と同様の方法で次の Theorem **??**を示すことができる. Theorem 2.2 では $\dim \mathrm{Bs}|L| \leq 0$ を仮定しているのに対して, Theorem **??**では $\mathrm{Bs}|L| = \emptyset$ を仮定していることに注意せよ.

Theorem 2.3 *Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Assume that $\mathrm{Bs}|L| = \emptyset$, $h^0(L) = n + m - 1$, where $m = g(X, L) - q(X)$. Let (M, A) be a reduction of (X, L). Then (X, L) is one of the following types:*

(1) *(X, L) is a hyperquadric fibration over a smooth projective curve with*

$$q(X) \leq \frac{3}{2} + \frac{2m-1}{2n}.$$

(2) *(X, L) is a classical scroll over a smooth projective surface Y with $\kappa(Y) = -\infty$. If $q(X) > 0$ and Y is relatively minimal, then*

$$q(X) \leq 1 + \frac{2m-n+1}{n^2-3n+4}.$$

If $q(X) > 0$ and Y is not relatively minimal, then

$$q(X) \leq 1 + \frac{4m-1}{8n^2-20n+16}.$$

(3) *$(M, A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ and (X, L) is obtained by 7 times simple blowing ups of (M, A).*

(4) *$(M, A) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$ and (X, L) is obtained by 7 times simple blowing ups of (M, A).*

(5) *$(M, A) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ and (X, L) is obtained by 8 times simple blowing ups of (M, A).*

(6) *M is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ with $(F, A|_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for any fiber F of it, and (X, L) is obtained by 7 times simple blowing ups of (M, A).*

(7) *(X, L) is a Mukai manifold with $L^n = 2m - 2$ and $\Delta(X, L) = m - 1$.*

Proof. [10] を参照のこと. □

3 Main results

いよいよ主結果を述べる.

Theorem 3.1 *Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Assume that L is very ample and $g(X, L) - q(X) = 3$. Then (X, L) is one of the following types:*

(1) *A hyperquadric fibration over $\mathbb{P}^1$. Then e, b and L^n are the following*

L^n	7	8	9	10	11	12
e	2	3	4	5	6	7
b	3	2	1	0	-1	-2

(2) *A classical scroll over $\mathbb{P}^2$. In this case, there exists an ample vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ such that $X = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$ and $L = H(\mathcal{E})$. Then $\mathcal{E}$ is one of the following types.*

(2.1) $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4}$.

(2.2) $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$.

(2.3) $T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$.

(2.4) $\mathrm{rank}\mathcal{E} = 2$ *and* $\mathcal{E}$ *has type* (1,3) *or* (2,2).

(3) *X is a quartic hypersurface in $\mathbb{P}^{n+1}$ and $L = \mathcal{O}_X(1)$.*

Theorem 3.2 *Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Assume that L is very ample and $g(X, L) - q(X) = 4$. Then (X, L) is one of the following types:*

(1) *A hyperquadric fibration over a smooth curve C and one of the following two types is satisfied.*

(1.1) $g(C) = 1$, $e = 8$, $b = -4$ *and* $n = 3$.

(1.2) $g(C) = 0$ *and* e, b *and* L^n *are the following*

L^n	7	8	9	10	11	12	13	14	15
e	2	3	4	5	6	7	8	9	10
b	3	2	1	0	-1	-2	-3	-4	-5

(2) *A classical scroll over a smooth surface S. In this case, there exists an ample vector bundle $\mathcal{E}$ on S such that $X = \mathbb{P}_S(\mathcal{E})$ and $L = H(\mathcal{E})$. Moreover $(S, \mathcal{E})$ satisfies one of the following three types.*

(2.1) *S is a quadric surface in $\mathbb{P}^3$ and $c_1(\mathcal{E}) = \mathcal{O}_S(3)$.*

(2.2) *S is a cubic surface in $\mathbb{P}^3$ and $c_1(\mathcal{E}) = \mathcal{O}_S(2)$.*

(2.3) $S = \mathbb{P}_C(\mathcal{F})$ and $c_1(\mathcal{E}) = 2H(\mathcal{F}) + p^*(B)$, where C is a smooth curve with $g(C) \leq 1$, $\mathcal{F}$ is a vector bundle of rank two on C, $p : \mathbb{P}_C(\mathcal{F}) \to C$ is the projection and B is a line bundle on C. In this case, the rank of $\mathcal{E}$ is two.

(3) X is a complete intersection of type $(2, 3)$ in $\mathbb{P}^{n+2}$ and $L = \mathcal{O}_X(1)$.

Theorem 3.3 *Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Assume that L is very ample and $g(X, L) - q(X) = 5$. Let (M, A) be a reduction of (X, L). Then (X, L) is one of the following types:*

(1) *A hyperquadric fibration over a smooth curve C and one of the following three types is satisfied.*

(1.1) $g(C) = 2$, $e = 8$, $b = -4$ and $n = 3$.

(1.1) $g(C) = 1$ and e, b and L^n are the following.

L^n	12	13	14	15
e	7	8	9	10
b	-2	-3	-4	-5

(1.2) $g(C) = 0$ and e, b and L^n are the following.

L^n	9	10	11	12	13	14	15	16	17	18
e	3	4	5	6	7	8	9	10	11	12
b	3	2	1	0	-1	-2	-3	-4	-5	-6

(2) *A classical scroll over a smooth surface S. In this case, there exists an ample vector bundle $\mathcal{E}$ on S such that $X = \mathbb{P}_S(\mathcal{E})$ and $L = H(\mathcal{E})$. Moreover $(S, \mathcal{E})$ satisfies one of the following two types.*

(2.1) $S = F_1$ and $c_1(\mathcal{E})^2 = 21$.

(2.2) $S = \mathbb{P}_C(\mathcal{F})$ and $c_1(\mathcal{E}) = 2H(\mathcal{F}) + p^*(B)$, where C is a smooth curve with $g(C) \leq 1$, $\mathcal{F}$ is a vector bundle of rank two on C, $p : \mathbb{P}_C(\mathcal{F}) \to C$ is the projection and B is a line bundle on C. In this case, the rank of $\mathcal{E}$ is two.

(3) $(M, A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.

(4) $(M, A) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.

(5) X is a complete intersection of type $(2, 2, 2)$ in $\mathbb{P}^{n+3}$ and $L = \mathcal{O}_X(1)$.

(6) X is a one point blowing up of the type $(2, 2, 2)$ complete intersection Y in $\mathbb{P}^{n+3}$ and $L = \pi^*(\mathcal{O}_Y(1)) - E$, where $\pi : X \to Y$ is the birational morphism and E is the exceptional divisor.

Proof. 上記の 3 つの定理は基本的に証明方法が同じであるので Theorem 3.3 の証明の流れを述べることにする[1].

まず, L は非常に豊富な因子であるので $h^0(L) \geq n+1$ が成り立つことに注意する.

(a) $h^0(L) \geq n+4$ のとき, Theorem 2.2 と Theorem **??**を用いると (X,L) のタイプは特別なものになることがわかる. あとは個別に詳しく調べる.

(b) $h^0(L) = n+3$ のとき, X は L による埋め込みで $\mathbb{P}^{n+2}$ 内の余次元 2 の部分多様体となる. いま $n \geq 3$ なので Barth-Lasen の定理 ([15, Theorem 3.2.1]) より $h^1(\mathcal{O}_X) = 0$ となることがわかる. したがって $g(X,L) = h^1(\mathcal{O}_X) + 5 = 5$ となることがわかる. ここで $\kappa(K_X + (n-2)L)$ の値により場合分けを行う.
(b.1) もし $\kappa(K_X + (n-2)L) \geq 0$ なら

$$5 = g(X,L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1} \geq 1 + \frac{1}{2}L^n$$

なので $L^n \leq 8$ がいえる. L は非常に豊富な因子よりこのような (X,L) には分類表がある ([11], [12], [14]). それを用いて分類を得ることができる.
(b.2) もし $\kappa(K_X + (n-2)L) = -\infty$ なら, 随伴束の理論より (X,L) は特別な場合となる (Theorem 2.1 を参照せよ) ので個別に調べる.

(c) $h^0(L) = n+2$ のとき, X は L による埋め込みで $\mathbb{P}^{n+1}$ 内の余次元 1 の部分多様体, つまり $\mathbb{P}^{n+1}$ の超曲面となる. ここで $d = L^n$ とおくと $g(X,L) = \frac{1}{2}(d-1)(d-2)$ となることがわかる. $g(X,L) = 5$ より $10 = (d-1)(d-2)$ がいえる. しかしこれを満たす整数 d は存在しない.

(d) $h^0(L) = n+1$ のとき, (X,L) は $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ と同型となる. しかしこのときは $g(X,L) = 0$ となり, 仮定に反する. □

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Department of Mathematics
Faculty of Science
Kochi University
Akebono-cho, Kochi 780-8520
Japan
E-mail: fukuma@kochi-u.ac.jp
http://www.math.kochi-u.ac.jp/fukuma/indexj.html