

Calculations of sectional Euler numbers and sectional Betti numbers of special polarized manifolds

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Version 4
October 21, 2011

1 Introduction

In this note, we will calculate the i th sectional Euler number $e_i(X, L)$ and the i th sectional Betti number $b_i(X, L)$ of some special polarized manifolds (X, L) . We also note that results in this note are useful for classifications of polarized manifolds (for example see [5]). At any time, we will update this note if we complete calculations of sectional Euler numbers and sectional Betti numbers of new example¹.

2 Preliminaries

Notation 2.1 Let (X, L) be a polarized manifold of dimension n . For every integers i and j with $0 \leq i \leq n$ and $0 \leq j \leq i$, we put

$$C_j^i(X, L) := \sum_{l=0}^j (-1)^l \binom{n-i+l-1}{l} c_{j-l}(X) L^l,$$

Definition 2.1 ([3]) Let (X, L) be a polarized manifold of dimension n , and let i and j be integers with $0 \leq j \leq i \leq n$.

(i) The i -th sectional Euler number $e_i(X, L)$ of (X, L) is defined by the following:

$$e_i(X, L) := C_i^i(X, L) L^{n-i}.$$

(ii) The i -th sectional Betti number $b_i(X, L)$ of (X, L) is defined by the following:

$$b_i(X, L) := \begin{cases} e_0(X, L) & \text{if } i = 0, \\ (-1)^i \left(e_i(X, L) - \sum_{j=0}^{i-1} 2(-1)^j h^j(X, \mathbb{C}) \right) & \text{if } 1 \leq i \leq n. \end{cases}$$

Remark 2.1 (i) For every integers i and j with $0 \leq j \leq i \leq n$, $e_i(X, L)$, $b_i(X, L)$ and $w_i^j(X, L)$ are integer (see [3]).

(ii) If $i = 0$, then $e_0(X, L) = b_0(X, L) = L^n$. If $i = n$, then $e_n(X, L) = e(X)$ and $b_n(X, L) = h^n(X, \mathbb{C})$.

¹If you find a mistake in this note, please let me know.

3 Calculations

Example 3.1 The case where (X, L) is $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

Then

$$e_i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = e(\mathbb{P}^i) = i + 1$$

and

$$b_i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = b(\mathbb{P}^i) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

Example 3.2 The case where (X, L) is $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

Then

$$b_n(\mathbb{Q}^n) = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

$$b_{n-1}(\mathbb{Q}^n) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd,} \end{cases}$$

$$b_i(\mathbb{Q}^n) = \begin{cases} 1, & \text{if } i \text{ is even with } i \leq n-2, \\ 0, & \text{if } i \text{ is odd with } i \leq n-2, \end{cases}$$

Hence

$$e_i(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)) = e_i(\mathbb{Q}^i) = \begin{cases} i + 2, & \text{if } i \text{ is even,} \\ i + 1, & \text{if } i \text{ is odd,} \end{cases}$$

and

$$b_i(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)) = (-1)^i \left(e_i(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)) - 2 \sum_{j=0}^{i-1} b_j(\mathbb{Q}^n) \right) = \begin{cases} 2, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

Example 3.3 The case where (X, L) is $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.

Set $H = \mathcal{O}_{\mathbb{P}^4}(1)$. Then $c_1(\mathbb{P}^4) = 5H$, $c_2(\mathbb{P}^4) = 10H^2$, $c_3(\mathbb{P}^4) = 10H^3$, $c_4(\mathbb{P}^4) = 5H^4 = 5$.

Hence

$$e_0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = (2H)^4 = 16,$$

$$e_1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = \sum_{l=0}^1 (-1)^l \binom{2+l}{l} c_{1-l}(X) (2H)^{3+l} = -8,$$

$$e_2(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = \sum_{l=0}^2 (-1)^l \binom{1+l}{l} c_{2-l}(X) (2H)^{2+l} = 8,$$

$$e_3(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = \sum_{l=0}^3 (-1)^l \binom{l}{l} c_{3-l}(X) (2H)^{1+l} = 4,$$

$$e_4(\mathbb{P}^4) = e(\mathbb{P}^4) = 5.$$

On the other hand, since

$$b_i(\mathbb{P}^4) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd,} \end{cases}$$

we have

$$b_0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = 16,$$

$$b_1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = 10,$$

$$b_2(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = 6,$$

$$b_3(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = 0,$$

$$b_4(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = 1.$$

Example 3.4 The case where (X, L) is $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.
Set $H = \mathcal{O}_{\mathbb{Q}^3}(1)$. Then $c_1(\mathbb{Q}^3) = 3H$, $c_2(\mathbb{Q}^3) = 10H^2$, $c_3(\mathbb{Q}^3) = 2H^3 = 4$.

Hence

$$\begin{aligned} e_0(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= (2H)^3 = 16, \\ e_1(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= \sum_{l=0}^1 (-1)^l \binom{1+l}{l} c_{1-l}(X) (2H)^{2+l} = -8, \\ e_2(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= \sum_{l=0}^2 (-1)^l \binom{l}{l} c_{2-l}(X) (2H)^{1+l} = 8, \\ e_3(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= e(\mathbb{Q}^3) = 4. \end{aligned}$$

On the other hand, since

$$b_i(\mathbb{Q}^3) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd,} \end{cases}$$

we have

$$\begin{aligned} b_0(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 16, \\ b_1(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 10, \\ b_2(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 6, \\ b_3(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 0. \end{aligned}$$

Example 3.5 The case where (X, L) is $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.
Set $H = \mathcal{O}_{\mathbb{P}^3}(1)$. Then $c_1(\mathbb{P}^3) = 4H$, $c_2(\mathbb{P}^3) = 6H^2$, $c_3(\mathbb{P}^3) = 4H^3$.

Hence

$$\begin{aligned} e_0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= (3H)^3 = 27, \\ e_1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= \sum_{l=0}^1 (-1)^l \binom{1+l}{l} c_{1-l}(X) (3H)^{2+l} = -18, \\ e_2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= \sum_{l=0}^2 (-1)^l c_{2-l}(X) (3H)^{1+l} = 9, \\ e_3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= e(\mathbb{P}^3) = 4. \end{aligned}$$

On the other hand, since

$$b_i(\mathbb{P}^3) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd,} \end{cases}$$

we have

$$\begin{aligned} b_0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 16, \\ b_1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 10, \\ b_2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 6, \\ b_3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 0. \end{aligned}$$

Example 3.6 The case where (X, L) is a Veronese fibration over a smooth curve C (see [2, (13.10)]).

Then there exists a vector bundle \mathcal{E} of rank three on C such that $X = \mathbb{P}_C(\mathcal{E})$ and $L = 2H(\mathcal{E}) + f^*(B)$, where $f: X \rightarrow C$ is its fibration and $B \in \text{Pic}(C)$. Set $e := \deg \mathcal{E}$ and $b := \deg B$. First we

calculate $e_i(X, L)$. Here we note that $2g(C) - 2 + e + 2b = 0$, $L^3 = 8e + 12b$ and $g_1(X, L) = 1 + 2e + 2b$. Then

$$e_0(X, L) = L^3 = 8e + 12b, e_1(X, L) = 2 - 2g_1(X, L) = -4e - 4b.$$

Next we calculate $e_2(X, L)$. Since

$$\begin{aligned} c_2(X) &= \sum_{j=0}^2 \sum_{k=0}^j \binom{3-k}{j-k} c_k(f^*(\mathcal{E}^\vee)) H(\mathcal{E})^{j-k} c_{j-k}(f^*(\mathcal{T}_C)) \\ &= 3c_1(f^*(\mathcal{T}_C))H(\mathcal{E}) + 3H(\mathcal{E})^2 + 2c_1(f^*(\mathcal{E}^\vee))H(\mathcal{E}), \end{aligned}$$

we have

$$\begin{aligned} e_2(X, L) &= \sum_{l=0}^2 (-1)^l \binom{2+l}{l} c_{2-l}(X) (2H + f^*(B))^{1+l} \\ &= 20e + 27b. \end{aligned}$$

Next we calculate $e_3(X, L)$. We note that $e_3(X, L) = e(X)$. Since

$$\begin{aligned} b_0(X) &= 1, \\ b_1(X) &= 2g(C), \\ b_2(X) &= 2, \\ b_3(X) &= 2g(C), \end{aligned}$$

we have $e_3(X, L) = e(X) = 6 - 6g(C) = 3e + 6b$.

Furthermore we calculate $b_i(X, L)$. Then

$$\begin{aligned} b_0(X, L) &= 8e + 12e, \\ b_1(X, L) &= 2(1 + 2e + 2b), \\ b_2(X, L) &= 19e + 25b, \\ b_3(X, L) &= 2 - e - 2b. \end{aligned}$$

Example 3.7 The case where (X, L) is a Del Pezzo manifold.

Here we note that by [2, (8.11) Theorem], we have $L^n \leq 8$ and (X, L) is one of the following:

(3.7.1) $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.

First we calculate $e_i(X, L)$. Since

$$\begin{aligned} e_i(X, L) &= \sum_{l=0}^i (-1)^l \binom{n-i+l-1}{l} c_{i-l}(X) L^{n-i+l} \\ &= \sum_{l=0}^i (-1)^l \binom{n-i+l-1}{l} \binom{n+1}{i-l} 2^{n-i+l}, \end{aligned}$$

we have

$$\begin{aligned} e_0(X, L) &= \left((-1)^0 \binom{2}{0} \binom{4}{0} 2^0 \right) 2^3 = 8, \\ e_1(X, L) &= \left((-1)^0 \binom{1}{0} \binom{4}{1} 2^0 + (-1)^1 \binom{2}{1} \binom{4}{0} 2^1 \right) 2^2 = 0, \\ e_2(X, L) &= \left((-1)^0 \binom{0}{0} \binom{4}{2} 2^0 + (-1)^1 \binom{1}{1} \binom{4}{1} 2^1 + (-1)^2 \binom{2}{2} \binom{4}{0} 2^2 \right) 2 = 4, \\ e_3(X, L) &= \left((-1)^0 \binom{-1}{0} \binom{4}{3} 2^0 + (-1)^1 \binom{0}{1} \binom{4}{2} 2^1 + (-1)^2 \binom{1}{2} \binom{4}{1} 2^2 + (-1)^3 \binom{2}{3} \binom{4}{0} 2^3 \right) 2^0 = 4. \end{aligned}$$

Next we calculate $b_i(X, L)$. Since

$$b_j(X, \mathbb{C}) = \begin{cases} 1, & \text{if } j = 0, 2, \\ 0, & \text{if } j = 1, 3, \end{cases}$$

we have

$$\begin{aligned} b_0(X, L) &= e_0(X, L) = 8, \\ b_1(X, L) &= -e_1(X, L) + 2b_0(X) = 2, \\ b_2(X, L) &= e_2(X, L) - 2(b_0(X) - b_1(X)) = 2, \\ b_3(X, L) &= -e_3(X, L) + 2(b_0(X) - b_1(X) + b_2(X)) = 0. \end{aligned}$$

(3.7.2) X is the blowing up of \mathbb{P}^3 at a point and $L = \pi^*(\mathcal{O}_{\mathbb{P}^3}(2)) - E$, where $\pi : X \rightarrow \mathbb{P}^3$ is its birational morphism and E is the exceptional divisor. Then by [3, Theorem 3.2] and (3.7.1) above, we have

$$\begin{aligned} e_0(X, L) &= 7, \\ e_1(X, L) &= 0, \\ e_2(X, L) &= 5, \\ e_3(X, L) &= 6. \end{aligned}$$

and

$$\begin{aligned} b_0(X, L) &= 7, \\ b_1(X, L) &= 2, \\ b_2(X, L) &= 3, \\ b_3(X, L) &= 0. \end{aligned}$$

(3.7.3) (X, L) is either

$$(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1)), (\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1)) \text{ or } (\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}), H(T_{\mathbb{P}^2}))$$

where p_i is the i th projection and $T_{\mathbb{P}^2}$ is the tangent bundle of \mathbb{P}^2 .

(3.7.3.1) The case where $(X, L) \cong (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1))$.

Since $\mathcal{T}_X \cong \oplus_{j=1}^3 p_j^* \mathcal{T}_{\mathbb{P}^1}$, we have

$$\begin{aligned} c_1(\mathcal{T}_X) &= \sum_{j=1}^3 p_j^* c_1(\mathcal{T}_{\mathbb{P}^1}) \\ &= \sum_{j=1}^3 p_j^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)), \\ c_2(\mathcal{T}_X) &= p_1^* c_1(\mathcal{T}_{\mathbb{P}^1}) p_2^* c_1(\mathcal{T}_{\mathbb{P}^1}) + p_1^* c_1(\mathcal{T}_{\mathbb{P}^1}) p_3^* c_1(\mathcal{T}_{\mathbb{P}^1}) + p_2^* c_1(\mathcal{T}_{\mathbb{P}^1}) p_3^* c_1(\mathcal{T}_{\mathbb{P}^1}) \\ &= p_1^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) p_2^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) + p_1^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) p_3^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) + p_2^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) p_3^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) \\ c_3(X) &= e(X). \end{aligned}$$

On the other hand

$$\begin{aligned} b_0(X) &= 1, \\ b_1(X) &= 0, \\ b_2(X) &= 3, \\ b_3(X) &= 0. \end{aligned}$$

Therefore

$$\begin{aligned}
e_0(X, L) &= L^3 = 6, \\
e_1(X, L) &= \sum_{l=0}^1 (-1)^l \binom{1+l}{l} c_{1-l}(X) L^{2+l} = 0, \\
e_2(X, L) &= \sum_{l=0}^2 (-1)^l \binom{l}{l} c_{2-l}(X) L^{1+l} = 6, \\
e_3(X, L) &= e(X) = 8,
\end{aligned}$$

and

$$\begin{aligned}
b_0(X, L) &= e_0(X, L) = 6, \\
b_1(X, L) &= -e_1(X, L) + 2b_0(X) = 2, \\
b_2(X, L) &= e_2(X, L) - 2(b_0(X) - b_1(X)) = 4, \\
b_3(X, L) &= -e_3(X, L) + 2(b_0(X) - b_1(X) + b_2(X)) = 0.
\end{aligned}$$

(3.7.3.2) The case where $(X, L) \cong (\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1))$.

Since $\mathcal{T}_X \cong \oplus_{j=1}^2 p_j^*(\mathcal{T}_{\mathbb{P}^2})$, we have

$$\begin{aligned}
c_1(\mathcal{T}_X) &= \sum_{j=1}^2 p_j^* c_1(\mathcal{T}_{\mathbb{P}^2}) \\
&= \sum_{j=1}^2 p_j^* c_1(\mathcal{O}_{\mathbb{P}^2}(3)), \\
c_2(\mathcal{T}_X) &= p_1^* c_2(\mathcal{T}_{\mathbb{P}^2}) + p_1^* c_1(\mathcal{T}_{\mathbb{P}^2}) p_2^* c_1(\mathcal{T}_{\mathbb{P}^2}) + p_2^* c_2(\mathcal{T}_{\mathbb{P}^2}) \\
&= 3p_1^* \mathcal{O}_{\mathbb{P}^2}(1)^2 + 9p_1^* \mathcal{O}_{\mathbb{P}^2}(1) p_2^* \mathcal{O}_{\mathbb{P}^2}(1) + 3p_2^* \mathcal{O}_{\mathbb{P}^2}(1)^2, \\
c_3(\mathcal{T}_X) &= p_1^* c_2(\mathcal{T}_{\mathbb{P}^2}) p_2^* c_1(\mathcal{T}_{\mathbb{P}^2}) + p_1^* c_1(\mathcal{T}_{\mathbb{P}^2}) p_2^* c_2(\mathcal{T}_{\mathbb{P}^2}) \\
&= 9p_1^* \mathcal{O}_{\mathbb{P}^2}(1)^2 p_2^* \mathcal{O}_{\mathbb{P}^2}(1) + 9p_1^* \mathcal{O}_{\mathbb{P}^2}(1) p_2^* \mathcal{O}_{\mathbb{P}^2}(1)^2, \\
c_4(X) &= e(X).
\end{aligned}$$

On the other hand

$$\begin{aligned}
b_0(X) &= 1, \\
b_1(X) &= 2q(X) = 0, \\
b_2(X) &= 2b_2(\mathbb{P}^2)b_0(\mathbb{P}^2) + b_1(\mathbb{P}^2)b_1(\mathbb{P}^2) = 2, \\
b_3(X) &= 2b_3(\mathbb{P}^2)b_0(\mathbb{P}^2) + 2b_2(\mathbb{P}^2)b_1(\mathbb{P}^2) = 0, \\
b_4(X) &= 2b_4(\mathbb{P}^2)b_0(\mathbb{P}^2) + 2b_3(\mathbb{P}^2)b_1(\mathbb{P}^2) + b_2(\mathbb{P}^2)b_2(\mathbb{P}^2) = 3.
\end{aligned}$$

Therefore

$$\begin{aligned}
e_0(X, L) &= L^4 = 6, \\
e_1(X, L) &= \sum_{l=0}^1 (-1)^l \binom{2+l}{l} c_{1-l}(X) L^{3+l} = 0, \\
e_2(X, L) &= \sum_{l=0}^2 (-1)^l \binom{1+l}{l} c_{2-l}(X) L^{2+l} = 6, \\
e_3(X, L) &= \sum_{l=0}^3 (-1)^l \binom{l}{l} c_{3-l}(X) L^{1+l} = 6, \\
e_4(X, L) &= e(X) = 9,
\end{aligned}$$

and

$$\begin{aligned}
b_0(X, L) &= e_0(X, L) = 6, \\
b_1(X, L) &= -e_1(X, L) + 2b_0(X) = 2, \\
b_2(X, L) &= e_2(X, L) - 2(b_0(X) - b_1(X)) = 4, \\
b_3(X, L) &= -e_3(X, L) + 2(b_0(X) - b_1(X) + b_2(X)) = 0, \\
b_4(X, L) &= e_4(X, L) - 2(b_0(X) - b_1(X) + b_2(X) - b_3(X)) = 3.
\end{aligned}$$

(3.7.3.3) The case where $(X, L) \cong (\mathbb{P}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2}), H(\mathcal{T}_{\mathbb{P}^2}))$.

First we note that

$$\begin{aligned}
b_0(X) &= 1, \\
b_1(X) &= 0, \\
b_2(X) &= 2, \\
b_3(X) &= 0.
\end{aligned}$$

Then by [4, Corollary 3.1 (3.1.2) and Corollary 3.3 (3.3.2)] we have

$$\begin{aligned}
e_0(X, L) &= s_2(\mathcal{T}_{\mathbb{P}^2}) = K_{\mathbb{P}^2}^2 - c_2(\mathbb{P}^2) = 6, \\
e_1(X, L) &= -(c_1(\mathcal{T}_{\mathbb{P}^2}) + K_{\mathbb{P}^2})c_1(\mathcal{T}_{\mathbb{P}^2}) = 0, \\
e_2(X, L) &= c_2(\mathbb{P}^2) + c_2(\mathcal{T}_{\mathbb{P}^2}) = 6, \\
e_3(X, L) &= 2e(\mathbb{P}^2) = 6,
\end{aligned}$$

and

$$\begin{aligned}
b_0(X, L) &= e_0(X, L) = 6, \\
b_1(X, L) &= (c_1(\mathcal{T}_{\mathbb{P}^2}) + K_{\mathbb{P}^2})c_1(\mathcal{T}_{\mathbb{P}^2}) + 2 = 2, \\
b_2(X, L) &= b_2(X) + c_2(\mathbb{P}^2) - 1 = 4, \\
b_3(X, L) &= b_3(X) = 0.
\end{aligned}$$

(3.7.4) The case where (X, L) is a linear section of the Grassmann variety $\text{Gr}(5, 2)$ parametrizing lines in \mathbb{P}^4 , embedded in \mathbb{P}^9 via the Plücker embedding. Then $L^n = 5$.

Remark 3.1 Here we review the Chern class of $\text{Gr}(p, q)$ parametrizing \mathbb{P}^{q-1} in \mathbb{P}^{p-1} (see [7, Chapter 14, 14.7]).

(i) Let S (resp. Q) be the universal subbundle (resp. the universal quotient bundle) of $\text{Gr}(p, q)$. Then

$$c(\text{Gr}(p, q)) = c(S^\vee \otimes Q). \quad (1)$$

We note that $\text{rank} S = q$ and $\text{rank} Q = p - q$. From (1),

$$ch(\mathcal{T}_{\text{Gr}(p, q)}) = ch(S^\vee)ch(Q) \quad (2)$$

holds. Since $ch(Q) + ch(S) = p$, we have

$$ch(S) = q - \sum_{k \geq 1} ch_k(Q).$$

On the other hand

$$ch_k(S^\vee) = q - \sum_{k \geq 1} (-1)^k ch_k(Q). \quad (3)$$

(ii) Let $A_0 \subsetneq A_1 \subsetneq \dots \subsetneq A_{q-1} \subset \mathbb{P}^{p-1}$ be a flag of subspaces with $a_i = \dim A_i$, and let

$$\Omega(A_0, \dots, A_{q-1}) = \{L \in \text{Gr}(p, q) \mid \dim(L \cap A_i) \geq i, 0 \leq i \leq q\}.$$

Then $\Omega(A_0, \dots, A_{q-1})$ is a subvariety of dimension $\sum_{i=0}^{q-1} (a_i - i)$, which is called a *Schubert variety*. Then we set $(a_0, \dots, a_{q-1}) = [\Omega(A_0, \dots, A_{q-1})]$.

(iii) Next we explain the Schubert calculus. For $\lambda = (\lambda_0, \dots, \lambda_{q-1})$ with $p - q \geq \lambda_0 \geq \dots \geq \lambda_{q-1} \geq 0$, we set

$$\{\lambda_0, \dots, \lambda_{q-1}\} = \det(c_{\lambda_i + j - i}(Q))_{0 \leq i, j \leq q-1}.$$

Then $c_m(Q) = \{m, 0, \dots, 0\}$. We note that the following equality holds.

$$\{\lambda\} \cdot c_m(Q) = \sum \{\mu\}, \quad (4)$$

where the sum over μ with $p - q \geq \mu_0 \geq \lambda_0 \geq \dots \geq \mu_{q-1} \geq \lambda_{q-1}$ and $\sum_{i=0}^{q-1} \lambda_i = -m + \sum_{i=0}^{q-1} \mu_i$.

Moreover we have

$$\int_{\text{Gr}(p, q)} c_1(Q)^k \{\lambda_0, \dots, \lambda_{q-1}\} \int_{\text{Gr}(p, q)} c_1(Q)^k (a_0, \dots, a_{q-1}) = \frac{k!}{a_0! \dots a_{q-1}!} \prod_{i < j} (a_j - a_i). \quad (5)$$

Here $a_i = p - q + i - \lambda_i$, $k = \sum_{i=0}^{q-1} a_i - \frac{(q-1)q}{2} = \dim \text{Gr}(p, q) - \sum_{i=0}^{q-1} \lambda_i = q(p - q) - \sum_{i=0}^{q-1} \lambda_i$.

Now we consider the case where $p = 5$ and $q = 2$. Then first we calculate $c_j(\text{Gr}(5, 2))$ for $1 \leq j \leq 5$. From (2) and (3), we have

$$\begin{aligned} ch(\text{Gr}(5, 2)) &= ch(S^\vee)ch(Q) \\ &= \left(2 - \sum_{k \geq 1} (-1)^k ch_k(Q)\right) \left(3 + \sum_{k \geq 1} ch_k(Q)\right). \end{aligned}$$

Using this, we get the following. (Here we note that $c_j(Q) = 0$ for $j \geq 4$ because $\text{rank} Q = 3$.)

$$\begin{aligned} c_1(\text{Gr}(5, 2)) &= 5c_1(Q) \\ c_2(\text{Gr}(5, 2)) &= 12c_1(Q)^2 - c_2(Q) \\ c_3(\text{Gr}(5, 2)) &= 20c_1(Q)^3 - 10c_1(Q)c_2(Q) + 5c_3(Q) \\ c_4(\text{Gr}(5, 2)) &= 28c_1(Q)^4 - 38c_1(Q)^2c_2(Q) + 20c_1(Q)c_3(Q) + 7c_2(Q)^2 \\ c_5(\text{Gr}(5, 2)) &= 36c_1(Q)^5 - 90c_1(Q)^3c_2(Q) + 40c_1(Q)^2c_3(Q) + 45c_1(Q)c_2(Q)^2 - 10c_2(Q)c_3(Q). \end{aligned}$$

Next we use the Schubert calculus. First from (5) we get the following.

$$\begin{aligned} c_1(Q)^6 &= 5, \\ c_1(Q)^4 c_2(Q) &= 3, \\ c_1(Q)^3 c_3(Q) &= 1. \end{aligned}$$

Next we calculate $c_2(Q)^2 c_1(Q)^2$. Since $\{2, 0\} \cdot \{2, 0\} = \{3, 1\} + \{2, 2\}$, we have

$$\begin{aligned} \int_{\text{Gr}(5, 2)} c_2(Q)^2 c_1(Q)^2 &= \int_{\text{Gr}(5, 2)} c_1(Q)^2 \{3, 1\} + \int_{\text{Gr}(5, 2)} c_1(Q)^2 \{2, 2\} \\ &= \int_{\text{Gr}(5, 2)} c_1(Q)^2 (0, 3) + \int_{\text{Gr}(5, 2)} c_1(Q)^2 (1, 2) \\ &= 2. \end{aligned}$$

Next we calculate $c_2(Q)c_3(Q)c_1(Q)$. Since $\{2, 0\} \cdot \{3, 0\} = \{3, 2\}$, we have

$$\begin{aligned} \int_{\text{Gr}(5,2)} c_2(Q)c_3(Q)c_1(Q) &= \int_{\text{Gr}(5,2)} c_1(Q)\{3, 2\} = \int_{\text{Gr}(5,2)} c_1(Q)(0, 2) \\ &= 1. \end{aligned}$$

Hence

$$\begin{aligned} c_1(\text{Gr}(5, 2))L^5 &= 5c_1(Q)^6 = 25 \\ c_2(\text{Gr}(5, 2))L^4 &= 12c_1(Q)^6 - c_1(Q)^4c_2(Q) = 57 \\ c_3(\text{Gr}(5, 2))L^3 &= 20c_1(Q)^6 - 10c_1(Q)^4c_2(Q) + 5c_1(Q)^3c_3(Q) = 75 \\ c_4(\text{Gr}(5, 2))L^2 &= 28c_1(Q)^6 - 38c_1(Q)^4c_2(Q) + 20c_1(Q)^3c_3(Q) + 7c_1(Q)^2c_2(Q)^2 = 60 \\ c_5(\text{Gr}(5, 2))L &= 36c_1(Q)^6 - 90c_1(Q)^4c_2(Q) + 40c_1(Q)^3c_3(Q) \\ &\quad + 45c_1(Q)^2c_2(Q)^2 - 10c_1(Q)c_2(Q)c_3(Q) = 30. \end{aligned}$$

Therefore

$$\begin{aligned} e_0(X, L) &= L^6 = 5, \\ e_1(X, L) &= c_1(X)L^5 - 5L^6 = 0, \\ e_2(X, L) &= c_2(X)L^4 - 4c_1(X)L^5 + 10L^6 = 7, \\ e_3(X, L) &= c_3(X)L^3 - 3c_2(X)L^4 + 6c_1(X)L^5 - 10L^6 = 4, \\ e_4(X, L) &= c_4(X)L^2 - 2c_3(X)L^3 + 3c_2(X)L^4 - 4c_1(X)L^5 + 5L^6 = 6, \\ e_5(X, L) &= c_5(X)L - c_4(X)L^2 + c_3(X)L^3 - c_2(X)L^4 + c_1(X)L^5 - L^6 = 8, \\ e_6(X, L) &= e(X) = 10. \end{aligned}$$

Next we calculate $b_i(X, L)$. Since $b_0(X) = b_2(X) = 1$, $b_4(X) = b_6(X) = b_8(X) = 2$, $b_{10}(X) = b_{12}(X) = 1$ and $b_j(X) = 0$ for every positive odd integer j , we have

$$\begin{aligned} b_0(X, L) &= 5, \\ b_1(X, L) &= 2, \\ b_2(X, L) &= 5, \\ b_3(X, L) &= 0, \\ b_4(X, L) &= 2, \\ b_5(X, L) &= 4, \\ b_6(X, L) &= 2. \end{aligned}$$

(3.7.5) The case where (X, L) is a complete intersection of two hyperquadrics in \mathbb{P}^{n+2} . Then $L^n = 4$. First we calculate $e(X)$ in this case. In general we can prove the following.

Lemma 3.1 *Let (X, L) be a complete intersection of two hypersurfaces of degree s and t in \mathbb{P}^{n+2} . Then*

$$e(X) = -\frac{s}{t^2}(1-t)^{n+3} \sum_{k=0}^{n-1} \left(\frac{s}{t}\right)^k + \frac{s}{t^2} \sum_{j=0}^{n-1} \left(\frac{s}{t}\right)^j \sum_{k=0}^{2+j} (-t)^k \binom{n+3}{n+3-k} + (-s)^{n+1}(-t).$$

Proof. Let $c_j := c_j(X)$ and $H := \mathcal{O}_X(1)$. Then the following holds (see [7, Example 3.2.12]).

$$(1 + H)^{n+3} = C(X)(1 + sH)(1 + tH).$$

Here $C(X) = (1 + c_1 + \cdots + c_n)$. Hence

$$\begin{aligned} (c_n + sc_{n-1}H) + t(c_{n-1}H + sc_{n-2}H^2) &= \binom{n+3}{n} H^n \\ (c_{n-1} + sc_{n-2}H) + t(c_{n-2}H + sc_{n-3}H^2) &= \binom{n+3}{n-1} H^{n-1} \\ &\vdots \\ (c_2 + sc_1H) + t(c_1H + sc_0H^2) &= \binom{n+3}{2} H^2 \end{aligned}$$

Hence

$$\begin{aligned} &c_n + sc_{n-1}H + (-t)^{n-2} \cdot t(c_1H^{n-1} + sc_0H^n) \\ &= \left(\binom{n+3}{n} + (-t) \binom{n+3}{n-1} + \cdots + (-t)^{n-3} \binom{n+3}{3} + (-t)^{n-2} \binom{n+3}{2} \right) H^n \end{aligned}$$

Moreover since $c_1H^{n-1} = \mathcal{O}(n-s-t+3)H^{n-1}$, we have $c_1H^{n-1} + sc_0H^n = (n-t+3)H^n$. Therefore

$$\begin{aligned} &c_n + sc_{n-1}H \\ &= \left(\binom{n+3}{n} + (-t) \binom{n+3}{n-1} + \cdots \right. \\ &\quad \left. \cdots + (-t)^{n-3} \binom{n+3}{3} + (-t)^{n-2} \binom{n+3}{2} + (-t)^{n-1} \binom{n+3}{1} + (-t)^n \right) H^n \\ &= -\frac{s}{t^2} \left((-t)^3 \binom{n+3}{n} + (-t)^4 \binom{n+3}{n-1} + \cdots \right. \\ &\quad \left. \cdots + (-t)^n \binom{n+3}{3} + (-t)^{n+1} \binom{n+3}{2} + (-t)^{n+2} \binom{n+3}{1} + (-t)^{n+3} \right) \\ &= -\frac{s}{t^2} \left((1-t)^{n+3} - 1 - (-t)^1 \binom{n+3}{n+2} - (-t)^2 \binom{n+3}{n+1} \right) \\ &= -\frac{s}{t^2} \left((1-t)^{n+3} - 1 + t \binom{n+3}{n+2} - t^2 \binom{n+3}{n+1} \right). \end{aligned}$$

By the same argument as above for every j with $1 \leq j \leq n-1$ we have

$$\begin{aligned} &c_j H^{n-j} + sc_{j-1} H^{n-j+1} \\ &= \frac{s}{(-t)^{n-j+2}} \left((1-t)^{n+3} - \sum_{k=0}^{n+2-j} (-t)^k \binom{n+3}{n+3-k} \right). \end{aligned}$$

Hence

$$c_n = -\frac{s}{t^2} (1-t)^{n+3} \sum_{k=0}^{n-1} \left(\frac{s}{t}\right)^k + \frac{s}{t^2} \sum_{j=0}^{n-1} \left(\frac{s}{t}\right)^j \sum_{k=0}^{2+j} (-t)^k \binom{n+3}{n+3-k} + (-s)^{n+1} (-t). \quad (6)$$

□

Lemma 3.2 *Let (X, L) be a complete intersection of two hyperquadrics in \mathbb{P}^{n+2} . Then*

$$e(X) = \begin{cases} 2n+4, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By Lemma 3.1 we have

$$c_n = (-2)^{n+2} + \frac{1}{2} \left(n(-1)^n + \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+3}{n+3-k} \right).$$

Next we prove the following.

Claim 3.1

$$\begin{aligned} & (-2)^{n+2} + \frac{1}{2} \left(n(-1)^n + \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+3}{n+3-k} \right) \\ &= \begin{cases} 0, & n \text{ is odd,} \\ 2n+4, & n \text{ is even.} \end{cases} \end{aligned} \quad (7)$$

Proof. First we note the following.

$$\begin{aligned} & \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+2}{n+2-k} \\ &= \sum_{j=1}^n \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + \sum_{j=1}^n (-2)^{n+2-j} \binom{n+2}{j} \\ &= \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + \sum_{k=0}^1 (-2)^k \binom{n+2}{n+2-k} + \sum_{j=1}^n (-2)^{n+2-j} \binom{n+2}{j}, \end{aligned} \quad (8)$$

$$\begin{aligned} \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+2}{n+3-k} &= \sum_{j=1}^n \sum_{k=1}^{n+2-j} (-2)^k \binom{n+2}{n+3-k} \\ &= \sum_{j=1}^n \sum_{k=0}^{n+1-j} (-2)^{k+1} \binom{n+2}{n+2-k} \\ &= \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^{k+1} \binom{n+2}{n+2-k} \\ &\quad + \sum_{k=0}^1 (-2)^{k+1} \binom{n+2}{n+2-k} \\ &= -2 \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} \\ &\quad + \sum_{k=0}^1 (-2)^{k+1} \binom{n+2}{n+2-k}. \end{aligned} \quad (9)$$

Then from (8) and (9) we have

$$\sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \left(\binom{n+2}{n+2-k} + \binom{n+2}{n+2-k} \right)$$

$$\begin{aligned}
&= -\sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} - \sum_{k=0}^1 (-2)^k \binom{n+2}{n+2-k} + \sum_{j=1}^n (-2)^{n+2-j} \binom{n+2}{j} \\
&= -\sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + 4n + 6 + (-1)^{n+2} - (-2)^{n+2}. \tag{10}
\end{aligned}$$

Here we prove (7) by induction on n .

If $n = 1$ and 2 , then (7) holds.

Next we assume that (7) holds for $n - 1$ is odd. Then by assumption we have the following equality.

$$(-2)^{n+1} + \frac{1}{2} \left((n-1)(-1)^{n-1} + \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} \right) = 0. \tag{11}$$

Then by using (11), we have

$$\begin{aligned}
&(-2)^{n+2} + \frac{1}{2} \left(n(-1)^n + \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+3}{n+3-k} \right) \\
&= (-2)^{n+2} + \frac{1}{2} \left(n(-1)^n - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + 4n + 6 \right. \\
&\quad \left. + (-1)^{n+2} - (-2)^{n+2} \right) \\
&= (-2)^{n+2} + \frac{1}{2} \left(n - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + 4n + 6 + (-1)^{n+2} - (-2)^{n+2} \right) \\
&= (-2)^{n+2} + \frac{1}{2} \left(n - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + 4n + 6 + 1 - (-2)^{n+2} \right) \\
&= (-2)^{n+2} + \frac{1}{2} (5n + 7 + 2(-2)^{n+1}(n-1)(-1)^{n-1}) \\
&= 2n + 4.
\end{aligned}$$

Next we assume that (7) holds for $n - 1$ is even. Then by assumption we have the following equality.

$$(-2)^{n+1} + \frac{1}{2} \left((n-1)(-1)^{n-1} + \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} \right) = 2n + 2. \tag{12}$$

Then by using (12), we have

$$\begin{aligned}
&(-2)^{n+2} + \frac{1}{2} \left(n(-1)^n + \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+3}{n+3-k} \right) \\
&= (-2)^{n+2} + \frac{1}{2} \left(-n - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + 4n + 6 + (-1)^{n+2} - (-2)^{n+2} \right)
\end{aligned}$$

$$\begin{aligned}
&= (-2)^{n+2} + \frac{1}{2} \left(3n + 5 - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} - (-2)^{n+2} \right) \\
&= (-2)^{n+2} + \frac{1}{2} (3n + 5 + 2(-2)^{n+1} + (n-1)(-1)^{n-1} - 2(2n+2) - (-2)^{n+2}) \\
&= 0.
\end{aligned}$$

This completes the proof of Claim 3.1. \square

From Claim 3.1 we get Lemma 3.2. \square

Remark 3.2 Let (X, L) be a complete intersection of two hypersurfaces of degree s and t in \mathbb{P}^{n+2} . Then from (6) we can write $e(X)$ as follows.

$$e(X) = (-1)^n st \left(\sum_{k=0}^n (-1)^k \binom{n+3}{k} \left(\sum_{j=0}^{n-k} s^{n-k-j} t^j \right) \right).$$

Proof.

$$\begin{aligned}
c_n &= -\frac{s}{t^2} (1-t)^{n+3} \left(1 + \frac{s}{t} + \cdots + \left(\frac{s}{t} \right)^{n-1} \right) + \frac{s}{t^2} \left\{ \left(1 + (-t) \binom{n+3}{n+2} + (-t)^2 \binom{n+3}{n+1} \right) \right. \\
&\quad \left. + \frac{s}{t} \left(1 + (-t) \binom{n+3}{n+2} + (-t)^2 \binom{n+3}{n+1} + (-t)^3 \binom{n+3}{n} \right) \right. \\
&\quad \left. + \cdots + \left(\frac{s}{t} \right)^{n-1} \left(1 + (-t) \binom{n+3}{n+2} + \cdots + (-t)^{n+1} \binom{n+3}{2} \right) \right\} + (-s)^{n+1} (-t) \\
&= -\frac{s}{t^2} \left((-t)^3 \binom{n+3}{n} + \cdots + (-t)^{n+3} \right) \left(1 + \frac{s}{t} + \cdots + \left(\frac{s}{t} \right)^{n-1} \right) \\
&\quad + \frac{s}{t^2} \left(\sum_{j=1}^{n-1} \left(\frac{s}{t} \right)^j \sum_{k=P_1}^j (-t)^{2+k} \binom{n+3}{n+1-k} \right) + (-s)^{n+1} (-t) \\
&= -\frac{s}{t^2} \left((-t)^3 \binom{n+3}{n} + \cdots + (-t)^{n+3} \right) - \frac{s^2}{t^3} \left((-t)^4 \binom{n+3}{n-1} + \cdots + (-t)^{n+3} \right) \\
&\quad - \frac{s^3}{t^4} \left((-t)^5 \binom{n+3}{n-2} + \cdots + (-t)^{n+3} \right) \cdots - \frac{s^n}{t^{n+1}} \left((-t)^{n+2} \binom{n+3}{1} + (-t)^{n+3} \right) \\
&\quad + (-s)^{n+1} (-t) \\
&= (-s) \left((-t) \binom{n+3}{n} + \cdots + (-t)^{n+1} \right) + s^2 \left((-t) \binom{n+3}{n-1} + \cdots + (-t)^n \right) \\
&\quad - s^3 \left((-t) \binom{n+3}{n-2} + \cdots + (-t)^{n-1} \right) \cdots + (-s)^n \left((-t) \binom{n+3}{1} + (-t)^2 \right) + (-s)^{n+1} (-t) \\
&= \sum_{j=1}^{n+1} (-s) (-t)^j \binom{n+3}{n+1-j} + \sum_{j=1}^n (-s)^2 (-t)^j \binom{n+3}{n-j} + \sum_{j=1}^{n-1} (-s)^3 (-t)^j \binom{n+3}{n-1-j} \\
&\quad + \cdots + \sum_{j=1}^2 (-s)^n (-t)^j \binom{n+3}{2-j} + \sum_{j=1}^1 (-s)^{n+1} (-t)^j \binom{n+3}{1-j} \\
&= st \left(\sum_{j=1}^{n+1} (-t)^{j-1} \binom{n+3}{n+1-j} + \sum_{j=1}^n (-s) (-t)^{j-1} \binom{n+3}{n-j} + \cdots + \sum_{j=1}^1 (-s)^n (-t)^{j-1} \binom{n+3}{1-j} \right)
\end{aligned}$$

$$\begin{aligned}
&= st \left(\sum_{j=0}^n (-t)^j \binom{n+3}{n-j} + \sum_{j=0}^{n-1} (-s)(-t)^j \binom{n+3}{n-1-j} + \cdots + \sum_{j=0}^0 (-s)^n (-t)^j \binom{n+3}{-j} \right) \\
&= st \left(\sum_{k=0}^n (-1)^{n-k} \binom{n+3}{k} \left(\sum_{j=0}^{n-k} s^{n-k-j} t^j \right) \right) \\
&= (-1)^n st \left(\sum_{k=0}^n (-1)^k \binom{n+3}{k} \left(\sum_{j=0}^{n-k} s^{n-k-j} t^j \right) \right).
\end{aligned}$$

So we get the assertion. \square

Here we go back to the case (3.7.5). In this case, there exists a smooth ladder $X \supset X_1 \supset \cdots \supset X_{n-1}$ of L such that (X_j, L_j) is complete intersection of two hyperquadrics in \mathbb{P}^{n-j+2} . Since $e_i(X, L) = e(X_{n-i})$, we see that

$$e_i(X, L) = \begin{cases} 2i + 4, & \text{if } i \text{ is even with } i \geq 2, \\ 0, & \text{if } i \text{ is odd with } i \geq 3. \end{cases}$$

We also note that

$$e_i(X, L) = \begin{cases} 4, & \text{if } i = 0, \\ 0, & \text{if } i = 1. \end{cases}$$

Next we calculate $b_i(X, L)$. Since

$$b_i(X) = \begin{cases} 1, & \text{if } i \text{ is even with } i \leq n-1, \\ 0, & \text{if } i \text{ is odd with } i \leq n-1, \end{cases}$$

we have

$$b_i(X, L) = \begin{cases} 2i + 4 - 2\frac{i}{2} = i + 4, & \text{if } i \text{ is even with } i \geq 2, \\ 0 + 2\frac{i+1}{2} = i + 1, & \text{if } i \text{ is odd with } i \geq 3. \end{cases}$$

We also note that

$$b_i(X, L) = \begin{cases} 4, & \text{if } i = 0, \\ 2, & \text{if } i = 1. \end{cases}$$

(3.7.6) The case where X is a hypercubic in \mathbb{P}^{n+1} and $L = \mathcal{O}_X(1)$.

Here we consider more general case than this. In general we can prove the following claim.

Lemma 3.3 *Let (X, L) be a polarized manifold of dimension n such that X is a hypersurface of degree m and $L = \mathcal{O}_X(1)$. Then*

$$\begin{aligned}
e_i(X, L) &= \frac{1}{m} \left((1-m)^{i+2} - 1 + m(i+2) \right), \\
b_i(X, L) &= \begin{cases} \frac{1}{m} \left((1-m)^{i+2} - 1 + m(i+2) \right) - i, & \text{if } i \text{ is even with } i \leq n-1, \\ -\frac{1}{m} \left((1-m)^{i+2} - 1 + m(i+2) \right) + i + 1, & \text{if } i \text{ is odd with } i \leq n-1. \end{cases}
\end{aligned}$$

Proof. First we calculate $e_n(X, L)$. Let $c_j := c_j(X)$ and $H := \mathcal{O}_X(1)$. Then the following holds (see [7, Example 3.2.12]).

$$(1 + H)^{n+2} = (1 + c_1 + \cdots + c_n)(1 + mH).$$

Hence

$$\begin{aligned}
c_n + mc_{n-1}H &= \binom{n+2}{n} H^n \\
c_{n-1} + mc_{n-2}H &= \binom{n+2}{n-1} H^{n-1} \\
&\vdots \\
c_1 + mH &= \binom{n+2}{1} H
\end{aligned}$$

So we have

$$\begin{aligned}
c_n &= (-m)^n H^n + \frac{1}{m^2} \left((-m)^2 \binom{n+2}{2} + (-m)^3 \binom{n+2}{3} + \cdots + (-m)^{n+1} \binom{n+2}{n+1} \right) H^n \\
&= m(-m)^n + \frac{1}{m^2} \left((1-m)^{n+2} - 1 - (-m) \binom{n+2}{1} - (-m)^{n+2} \right) m \\
&= \frac{1}{m} \left((1-m)^{n+2} - 1 + m(n+2) \right).
\end{aligned}$$

On the other hand, in this case, there exists a smooth ladder $X \supset X_1 \supset \cdots \supset X_{n-1}$ of L such that (X_j, L_j) is a hypersurface of degree m in \mathbb{P}^{n-j+1} . Since $e_i(X, L) = e(X_{n-i})$, by the above argument we see that

$$e_i(X, L) = \frac{1}{m} \left((1-m)^{i+2} - 1 + m(i+2) \right).$$

Next we calculate $b_i(X, L)$. Since

$$b_i(X) = \begin{cases} 1, & \text{if } i \text{ is even with } i \leq n-1, \\ 0, & \text{if } i \text{ is odd with } i \leq n-1, \end{cases}$$

we have

$$\begin{aligned}
b_i(X, L) &= \begin{cases} \frac{1}{m} \left((1-m)^{i+2} - 1 + m(i+2) \right) - 2 \cdot \frac{i}{2}, & \text{if } i \text{ is even with } i \leq n-1, \\ -\frac{1}{m} \left((1-m)^{i+2} - 1 + m(i+2) \right) + 2 \cdot \frac{i+1}{2}, & \text{if } i \text{ is odd with } i \leq n-1. \end{cases} \\
&= \begin{cases} \frac{1}{m} \left((1-m)^{i+2} - 1 + m(i+2) \right) - i, & \text{if } i \text{ is even with } i \leq n-1, \\ -\frac{1}{m} \left((1-m)^{i+2} - 1 + m(i+2) \right) + i + 1, & \text{if } i \text{ is odd with } i \leq n-1. \end{cases}
\end{aligned}$$

This completes the proof of Lemma 3.3. \square

(3.7.7) The case where X is a double covering of \mathbb{P}^n branched along a smooth hypersurface of degree 4, and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$. Here we consider more general case than this.

Lemma 3.4 *Let X be a double covering of \mathbb{P}^n branched along a smooth hypersurface of degree m with even $m \geq 4$, and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$. Then*

$$\begin{aligned}
e_i(X, L) &= i + 2 - \frac{1}{m} (m - 1 + (1 - m)^{i+1}), \\
b_i(X, L) &= \left(i + 2 - \frac{1}{m} (m - 1 + (1 - m)^{i+1}) \right) + (-1)^{i+1} \begin{cases} i & \text{if } i \text{ is even,} \\ i + 1 & \text{if } i \text{ is odd.} \end{cases}
\end{aligned}$$

Proof. First we calculate $e_n(X, L)$. Let B be the branch locus. Then

$$e(X) = 2e(\mathbb{P}^n) - e(B).$$

Hence by Lemma 3.3

$$\begin{aligned} e_n(X, L) &= e(X) \\ &= 2e(\mathbb{P}^n) - e(B) \\ &= 2n + 2 - \frac{1}{m} \left((1-m)^{n+1} + m(n+1) - 1 \right) \\ &= n + 2 - \frac{1}{m} (m - 1 + (1-m)^{n+1}). \end{aligned}$$

Next we consider $e_i(X, L)$. First we note that $\Delta(X, L) = 1$ in this case. Since $\text{Bs}|L| = \emptyset$, there exists a smooth ladder $X \supset X_1 \supset \cdots \supset X_{n-1}$ of L . Then we see that $\Delta(X_j, L_j) = 1$ and $L_j^{n-j} = 2$, where $L_j := L|_{X_j}$ because $g(X, L) = m/2 - 1 \geq 1 = \Delta(X, L)$ and $L^n = 2 = 2\Delta(X, L)$. Hence X_j is a double covering of \mathbb{P}^{n-j} branched along a smooth hypersurface of degree 4, and L_j is the pull-back of $\mathcal{O}_{\mathbb{P}^{n-j}}(1)$. Since $e_i(X, L) = e(X_{n-i})$, by the above argument we see that for every integer i with $i \geq 1$, we have

$$e_i(X, L) = i + 2 - \frac{1}{m} (m - 1 + (1-m)^{i+1}). \quad (13)$$

Here we note that $e_0(X, L) = L^n = 2 = 0 + 2 - \frac{1}{4} (3 + (-3)^{0+1})$. Hence (13) also holds for $i = 0$.

Next we calculate $b_i(X, L)$. By the Barth-type theorem (see e.g. [8, Theorem 7.1.15]), we have

$$b_i(X) = \begin{cases} 1, & \text{if } i \text{ is even with } i \leq n-1, \\ 0, & \text{if } i \text{ is odd with } i \leq n-1. \end{cases}$$

Hence we have

$$\begin{aligned} b_i(X, L) &= (-1)^i \left(e_i(X, L) - 2 \sum_{j=0}^{i-1} (-1)^j b_j(X) \right) \\ &= (-1)^i \left(i + 2 - \frac{1}{m} (m - 1 + (1-m)^{i+1}) \right) - 2(-1)^i \cdot \begin{cases} \frac{i-1+1}{2} & \text{if } i \text{ is even,} \\ \frac{i-1}{2} + 1 & \text{if } i \text{ is odd,} \end{cases} \\ &= \left(i + 2 - \frac{1}{m} (m - 1 + (1-m)^{i+1}) \right) + (-1)^{i+1} \begin{cases} i & \text{if } i \text{ is even,} \\ i + 1 & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

We get the assertion of Lemma 3.4. \square

(3.7.8) The case where (X, L) is a weighted hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, \dots, 1)$. Then $L^n = 1$ and $\text{Bs}|L| = \{p\}$ (see [1, (16.7) Theorem and Appendix 1]).

In this case, there exists a smooth ladder $X \supset X_1 \supset \cdots \supset X_{n-1}$ of L such that (X_j, L_j) is a weighted hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, \dots, 1)$. Since $e_i(X, L) = e(X_{n-i})$, in order to calculate $e_i(X, L)$ for $i \geq 1$, it suffices to calculate $e(X)$.

Let $\pi : X^* \rightarrow X$ be the blowing up at $p \in X$. Then $\pi^*(L) - E$ is base point free and let $f : X^* \rightarrow \mathbb{P}^{n-1}$ be the morphism defined by $|\pi^*(L) - E|$. In this case, there exists a projective bundle $p : V \rightarrow \mathbb{P}^{n-1}$ and a double covering $\rho : X^* \rightarrow V$ such that $f = p \circ \rho$. Here we note that $V = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$. Let H_V be the tautological line bundle of V and let B

be the branch locus of ρ . Then there exist $B_1 \in |H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)|$ and $B_2 \in |3H_V|$ such that $B_1 \cong \mathbb{P}^{n-1}$ and $B = B_1 + B_2$. Here we note that the following equality holds.

$$e(X) = e(X^*) - e(E) + 1, \quad (14)$$

$$e(X^*) = 2e(V) - e(B), \quad (15)$$

$$e(B) = e(B_1) + e(B_2). \quad (16)$$

Therefore in order to calculate $e(X)$, we need the value of $e(E)$, $e(B_1)$, $e(B_2)$, and $e(V)$.

First we note that

$$e(E) = e(\mathbb{P}^{n-1}) = n \quad (17)$$

and

$$e(B_1) = e(\mathbb{P}^{n-1}) = n. \quad (18)$$

Next we calculate $e(V)$. By [1, Proof of Lemma in Appendix 2], we see that there exist the following three exact sequence:

$$0 \rightarrow 2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1) \rightarrow \mathcal{T}_V \rightarrow \mathcal{T}_{\mathbb{P}^{n-1}}|_V \rightarrow 0, \quad (19)$$

$$0 \rightarrow \mathcal{O}_V \rightarrow H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))^\vee \otimes \pi^*(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) \rightarrow \mathcal{T}_{\mathbb{P}^{n-1}}|_V \rightarrow 0, \quad (20)$$

$$0 \rightarrow \mathcal{T}_{B_2} \rightarrow \mathcal{T}_V|_{B_2} \rightarrow (3H_V)|_{B_2} \rightarrow 0. \quad (21)$$

From (19), we have

$$c(\mathcal{T}_V) = c(2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))c(\mathcal{T}_{\mathbb{P}^{n-1}}|_V). \quad (22)$$

Hence

$$\begin{aligned} c_n(V) &= (2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))c_{n-1}(\mathcal{T}_{\mathbb{P}^{n-1}}|_V) \\ &= (2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))(n(\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{n-1}). \end{aligned}$$

By (20), we get

$$c((\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{\oplus n}) = c(\mathcal{O}_V)c(\pi^*\mathcal{T}_{\mathbb{P}^{n-1}}). \quad (23)$$

Hence

$$c_{n-1}(\pi^*\mathcal{T}_{\mathbb{P}^{n-1}}) = \binom{n}{n-1} \pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n-1}.$$

Therefore

$$\begin{aligned} c_n(V) &= (2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))(n(\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{n-1}) \\ &= 2nH_V\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n-1} \\ &= 2n. \end{aligned} \quad (24)$$

Next we calculate $e(B_2)$. Before this, we note the following. Let $\mathcal{E} := \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}$ and let $H(\mathcal{E})$ be the tautological line bundle of $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{E})$. Then $V = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{E})$ and $H(\mathcal{E}) = H_V$. In this case, since $c_j(\mathcal{E}) = 0$ for any $j \geq 2$, we have $s_j(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^{n-1}}(2)^j$. Therefore

$$H_V^j \pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n-j} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n-j} s_{j-1}(\mathcal{E}) = 2^{j-1}. \quad (25)$$

From (21), we have

$$c(\mathcal{T}|_{B_2}) = c(\mathcal{T}_{B_2})c(3H_V|_{B_2}). \quad (26)$$

From (26) we obtain the following:

$$\begin{aligned}
c_{n-1}(B_2) + c_{n-2}(B_2)(3H_V|_{B_2}) &= c_{n-1}(V)B_2 \\
c_{n-2}(B_2) + c_{n-3}(B_2)(3H_V|_{B_2}) &= c_{n-2}(V)B_2 \\
&\vdots \\
c_1(B_2) + 3H_V|_{B_2} &= c_1(V)B_2
\end{aligned}$$

Therefore

$$\begin{aligned}
c_{n-1}(B_2) &= 3(c_{n-1}(V)H_V + (-3)c_{n-2}(V)H_V^2 \\
&\quad + \cdots + (-3)^{n-2}c_1(V)H^{n-1} + (-3)^{n-1}H^n). \tag{27}
\end{aligned}$$

On the other hand, by (22) we have

$$\begin{aligned}
c_j(V) &= (2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))c_{j-1}(\mathcal{T}_{\mathbb{P}^{n-1}}|_V) + c_j(\mathcal{T}_{\mathbb{P}^{n-1}}|_V) \\
&= 2\binom{n}{j-1}H_V\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{j-1} + \left(\binom{n}{j} - 2\binom{n}{j-1}\right)\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^j.
\end{aligned}$$

Hence by using (25) we get

$$\begin{aligned}
c_j(V)H^{n-j} &= 2\binom{n}{j-1}H_V^{n-j+1}\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{j-1} + \left(\binom{n}{j} - 2\binom{n}{j-1}\right)H_V^{n-j}\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^j \\
&= 2^{n-j+1}\binom{n}{j-1} + 2^{n-j-1}\left(\binom{n}{j} - 2\binom{n}{j-1}\right) \\
&= 2^{n-j}\binom{n}{j-1} + 2^{n-j-1}\binom{n}{j}.
\end{aligned}$$

Therefore

$$\sum_{j=1}^{n-1}(-3)^{n-j-1}c_j(V)H_V^{n-j} = 2\sum_{j=1}^{n-1}(-6)^{n-j-1}\binom{n}{j-1} + \sum_{j=1}^{n-1}(-6)^{n-j-1}\binom{n}{j}. \tag{28}$$

On the other hand

$$\begin{aligned}
2\sum_{j=1}^{n-1}(-6)^{n-j-1}\binom{n}{j-1} &= \frac{1}{18}\sum_{j=1}^{n-1}(-6)^{n-j+1}\binom{n}{j-1} \\
&= \frac{1}{18}((1 + (-6))^n - (-6)n - 1) \\
&= \frac{1}{18}((-5)^n + 6n - 1),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^{n-1}(-6)^{n-j-1}\binom{n}{j} &= -\frac{1}{6}\sum_{j=1}^{n-1}(-6)^{n-j}\binom{n}{j} \\
&= -\frac{1}{6}((1 + (-6))^n - (-6)^n - 1) \\
&= \frac{1}{6}((-6)^n - (-5)^n + 1).
\end{aligned}$$

Since $H_V^n = 2^{n-1}$, from (27) we get

$$\begin{aligned} & c_{n-1}(B_2) \\ &= 3 \left(\frac{1}{18}((-5)^n + 6n - 1) + \frac{1}{6}((-6)^n - (-5)^n + 1) + (-3)^{n-1}2^{n-1} \right) \\ &= -\frac{1}{3}(-5)^n + \frac{3n+1}{3}. \end{aligned} \tag{29}$$

From (18) and (29), we have

$$e(B) = e(B_1) + e(B_2) = 2n + \frac{1 - (-5)^n}{3}. \tag{30}$$

By (24) and (30) we get

$$e(X^*) = 2e(V) - e(B) = 2n + \frac{(-5)^n - 1}{3}. \tag{31}$$

Therefore by (17) and (31)

$$e(X) = e(X^*) - e(E) + 1 = n + \frac{(-5)^n + 2}{3}. \tag{32}$$

So we see that

$$e_i(X, L) = i + \frac{(-5)^i + 2}{3} \tag{33}$$

for every integer i with $1 \leq i \leq n$. Here we note that this equality holds for the case where $i = 0$.

Next we calculate $b_i(X, L)$. Since we see from [1, (16.6) 4)] that

$$b_j(X) = \begin{cases} 1, & \text{if } j \text{ is even with } j \leq n-1, \\ 0, & \text{if } j \text{ is odd with } j \leq n-1, \end{cases}$$

we have

$$\begin{aligned} b_i(X, L) &= (-1)^i \left(e_i(X, L) - 2 \sum_{j=0}^{i-1} (-1)^j b_j(X) \right) \\ &= (-1)^i \left(i + \frac{(-5)^i + 2}{3} \right) - 2(-1)^i \cdot \begin{cases} \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i+1}{2} + 1, & \text{if } i \text{ is odd,} \end{cases} \\ &= \begin{cases} (-1)^i \frac{(-5)^i + 2}{3}, & \text{if } i \text{ is even,} \\ (-1)^i \frac{(-5)^i - 1}{3}, & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

Example 3.8 The case where (X, L) is a hyperquadric fibration over a smooth curve C . Let $f : X \rightarrow C$ be its morphism. We put $\mathcal{E} := f_*(L)$. Then \mathcal{E} is a locally free sheaf of rank $n+1$ on C . Let $\pi : \mathbb{P}_C(\mathcal{E}) \rightarrow C$ be the projection. Then there exists an embedding $i : X \hookrightarrow \mathbb{P}_C(\mathcal{E})$ such that $f = \pi \circ i$, $X \in |2H(\mathcal{E}) + \pi^*(B)|$ for some $B \in \text{Pic}(C)$ and $L = H(\mathcal{E})|_X$. Let $e := \deg \mathcal{E}$ and $b := \deg B$. Then by [6, Theorem 3.1], we see that the following holds. Let (X, L) be a hyperquadric fibration over a smooth curve C with $\dim X = n \geq 3$, and let i be an integer with $0 \leq i \leq n$. Then

$$e_i(X, L) = (-1)^i (2e + (i+1)b) + \begin{cases} 2(i+1)(1-g(C)) & \text{if } i \text{ is odd,} \\ 2i(1-g(C)) & \text{if } i \text{ is even.} \end{cases}$$

Example 3.9 The case where (X, L) is a scroll over a smooth curve C .

Then by [4, Corollary 3.1 (3.1.1) and Corollary 3.3 (3.3.1)], we see that the following holds. Let \mathcal{E} be an ample vector bundle of rank n on C such that $X = \mathbb{P}_C(\mathcal{E})$ and $L = H(\mathcal{E})$.

$$e_i(X, L) = \begin{cases} i(2 - 2g(C)) & \text{if } i \geq 1, \\ \deg \mathcal{E} & \text{if } i = 0. \end{cases}$$

$$b_i(X, L) = \begin{cases} h^i(X, \mathbb{C}) & \text{if } i \geq 1, \\ \deg \mathcal{E} & \text{if } i = 0. \end{cases}$$

Example 3.10 The case where (X, L) is a scroll over a smooth surface S .

Let \mathcal{E} be an ample vector bundle of rank $n - 1$ on S such that $X = \mathbb{P}_S(\mathcal{E})$ and $L = H(\mathcal{E})$. Then by [4, Corollary 3.1 (3.1.2) and Corollary 3.3 (3.3.2)], we see that the following holds.

$$e_i(X, L) = \begin{cases} (i - 1)c_2(S) & \text{if } i \geq 3, \\ c_2(S) + c_2(\mathcal{E}) & \text{if } i = 2, \\ -(c_1(\mathcal{E}) + K_S)c_1(\mathcal{E}) & \text{if } i = 1, \\ s_2(\mathcal{E}) & \text{if } i = 0. \end{cases}$$

$$b_i(X, L) = \begin{cases} h^i(X, \mathbb{C}) & \text{if } m \geq i \geq 3, \\ h^2(X, \mathbb{C}) + c_2(\mathcal{E}) - 1 & \text{if } i = 2, \\ c_1(\mathcal{E})(c_1(\mathcal{E}) + K_S) + 2 & \text{if } i = 1, \\ s_2(\mathcal{E}) & \text{if } i = 0. \end{cases}$$

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