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REFLEXIVE MODULES OF RANK ONE OVER WEYL ALGEBRAS OF NON-ZERO CHARACTERISTICS

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Abstract. Much of the properties of a Weyl algebra A_n over a base field of non-zero characteristic are explained in terms of connections and curvatures on a vector bundle on an affine space $X = \mathbb{A}^{2n}$. In particular, it is known that an algebra endomorphism φ of A_n gives rise to a symplectic endomorphism f of X with a gauge transformation g. In this paper we study converse problem of finding φ from an arbitrary symplectic endomorphism f of $X = \mathbb{A}^{2n}$. It is shown that given such f, we may construct a projective left A_n-module (which corresponds to "the sheaf of local gauge transformations") such that its triviality is equivalent to the existence of the "lift" φ. Some properties of such a module will be discussed using the theory of reflexive sheaves.

1. Introduction

Let $A = A_n(k)$ be a Weyl algebra over a field k of characteristic $p \neq 0$. The author has already shown ([12]) that each endomorphism of A with a degree $\leq \frac{p-1}{2}$ gives rise to a "shadow" symplectic endomorphism of $\mathbb{A}^{2n}$. The procedure there is further sorted out in [13]. It turned out that the key differential equation obtained in [12] is actually related to integrability of the sheaf of intertwiners. In this paper we continue the argument developed there and see in Proposition 2.5 that for each symplectic endomorphism f of $\mathbb{A}^{2n}$ of degree $\leq \frac{p-1}{2}$, there exists a projective left A-module $W^{(f)}$ of rank 1 such that its triviality is equivalent to the existence of a lift of f to an endomorphism of A.

Then a question arises whether the module $W^{(f)}$ is always trivial. To answer it, we need to give examples of projective A_n-modules of rank 1. Such modules are studied by J. T. Stafford (See, for example, [8],[9]) and other people. A notable result is a classification of $A_1(\mathbb{C})$-modules [3]. We may then ask (Problem 2.7): if such a module is homotopic to trivial module, then is it trivial? Our definition of homotopy of A_n-modules W_t is that it can be considered as an $A_{n+1} = A_n\langle \xi_{n+1}, \eta_{n+1}\rangle$-module with an identification $\xi_{n+1} = t$. (See Definition 2.8 for precise definition.)

We then broaden our outlook and deal with reflexive A-modules. It is known that reflexive A-modules of rank 1 is an intersection of two principal ideals of A. (We give an alternative proof of the result(Theorem 5.4) to keep us somewhat self-contained). Thus we may easily find examples of reflexive modules.

We proceed to see if they are projective. Arguments on norms developed in section 4, along with the arguments in connections, curvatures, and p-curvatures Theorem 2.4 are in action here.

As an illustration of our theory, we give in section 6 an example of a locally free A_2-module which gives a homotopy between a non trivial A_1-left module and a trivial one. This gives a negative answer to the problem 2.7.

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2. Preliminaries and notations.

In this section we give some preliminaries on Weyl algebras. Details are found for example in Author's paper [13],[12],[11].

Definition 2.1. Let n be a positive integer. A **Weyl algebra** $A_n(k)$ over a commutative ring k is an algebra over k generated by $2n$ elements $\{\gamma_1, \gamma_2, \dots, \gamma_{2n}\}$ with the "canonical commutation relations"

$$[\gamma_i, \gamma_j](= \gamma_i\gamma_j - \gamma_j\gamma_i) = h_{ij} \qquad (1 \le i, j \le 2n), \tag{CCR}$$

where h is a non-degenerate anti-Hermitian $2n \times 2n$ matrix of the following form:

$$(h_{ij}) = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

It may sometimes be convenient to give an alias to each of the generators:

$$\xi_1 = \gamma_1,\ \xi_2 = \gamma_2,\ \xi_3 = \gamma_3,\ \dots, \xi_n = \gamma_n,$$
$$\eta_1 = \gamma_{n+1},\ \eta_2 = \gamma_{n+2},\ \eta_3 = \gamma_{n+3},\ \dots, \eta_n = \gamma_{2n}.$$

The relation (CCR) then reads:

$$[\eta_j, \xi_i] = \begin{cases} 1 & \text{if } i = j. \\ 0 & \text{otherwise.} \end{cases}$$
$$[\xi_i, \xi_j] = 0, \quad [\eta_i, \eta_j] = 0.$$

In what follows we will denote by k a perfect field of characteristic $p \neq 0$. We will then denote by $A = A_n = A_n(k)$ the Weyl algebra, and by $R = Z(A)$ its center [1]. It is easy to see that R is equal to the polynomial algebra in $2n$-variables $\gamma_1^p, \dots, \gamma_{2n}^p$.

2.1. **A note on a Frobenius map.** Before explaining further, we give some note on a Frobenius map and fix some notations. Let $Y = \mathrm{Spec}(k)$ be a base scheme, $X = \mathrm{Spec}(R) = \mathrm{Spec}(k[\gamma_1^p, \dots, \gamma_{2n}^p])$ be an affine space over it. In a paper of Illusie [6], there is given a definition of relative Frobenius map (in a more general setting– we make use of a very special case of his). The definition may be summarized in the following diagram.

$$\begin{tikzcd} X & X^{(p)} = F_Y^* X \arrow[l] \arrow[d] & X \arrow[l, "F_{X/Y}"'] \arrow[d] \\ Y \arrow[u, leftarrow] & Y \arrow[l, "F_Y"'] & Y \arrow[l, equal] \end{tikzcd}$$

Since we have assumed k to be a perfect field, the Frobenius map $F_Y : Y \to Y$ is invertible. So we may pull back the right square of the diagram above by F_Y^{-1} and obtain the following commutative diagram.

$$\begin{tikzcd} X \arrow[d] & (F_Y^{-1})^* X = X^{(1/p)} \arrow[l, "\overline{F}_{X/Y}"'] \arrow[d] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

[1] Please pay attention. Unlike the preceding papers of the author, we use the letter R instead of the letter Z to denote the center of A.

where $\overline{F}_{X/Y} = (F_Y^{-1})^*(F_{X/Y})$. On the other hand, $F_{X/Y}$ gives rise to a bijection between the base spaces X and $X^{(p)}$. Thus we see that $\overline{F_{X/Y}}$ gives rise to a bijection

$$|\overline{F_{X/Y}}| : |X^{(1/p)}| \to |X|$$

between the base spaces. We will therefore identify these two spaces and regard quasi coherent sheaves on X and $X^{(1/p)}$ as sheaves on the same base space $|X|$. The structure sheaf of $X^{(1/p)}$ will then be denoted as $\mathcal{O}^{(1/p)}$. The structure sheaf $\mathcal{O}$ on X may be identified with a subsheaf of $\mathcal{O}^{(1/p)}$ via the pullback homomorphism $\overline{F}^*_{X/Y}$.

To going to and fro between modules and sheaves, we employ the following convention.

Convention 2.2. We denote by A_X the sheaf $A \otimes_R \mathcal{O}$ on X. More generally, for any R-module M and for any open set U of X, we denote by M_U the sheaf $A \otimes_R \mathcal{O}_U$ on U.

2.2. **An connection ∇^A associated with the Weyl algebra A.** We denote by $R^{(1/p)}$ the affine coordinate ring $\mathcal{O}_X^{(1/p)}(X)$ of $X^{(1/p)}$. The ring $R^{(1/p)}$ is an polynomial in $2n$ generators $T_1, \dots, T_{2n}$ where $T_j = (\gamma_j^p)^{1/p}$.

We have a matrix representation

$$\Phi_0 : A \to M_{p^n}(R^{(1/p)})$$

such that each $\Phi_0(\gamma_j)$ is decomposed into a sum

$$\gamma_j = T_j + \mu_j \tag{2.2.1}$$

of its (unique) eigen value and a nilpotent part μ_j. (One of the ways to describe the matrices $\{\mu_j\}$ is to use their "differential representation" on $V = k[x_1, x_2, \dots, x_n]/(x_1^p, x_2^p, \dots, x_n^p)$. Namely,

$$\mu_j.f = x_j f, \quad \mu_{n+j}.f = \frac{\partial}{\partial x_j} f \qquad (j = 1, 2, \dots, n).)$$

It is easy to see that the representation Φ_0 gives rise to an isomorphism

$$A \otimes_R R^{(1/p)} \cong M_{p^n}(R^{(1/p)}).$$

of $R^{(1/p)}$ algebras. We also note that there is a connection ∇^A on $M_{p^n}(R^{(1/p)})$ whose curvature and p-curvature are zero such that the set of the parallel sections

$$M_{p^n}(R^{(1/p)})^{\nabla^A}$$

is equal to $\Phi_0(A)$. In precise, ∇^A is defined by using a matrix valued function

$$F = \sum_{i,j=1}^{2n} \bar{h}_{ij} \mu_j T_i \tag{2.2.2}$$

(where $\bar{h}_{ij}$ is the ij-component of the inverse matrix $\bar{h}$ of h.) as follows.

$$\nabla^A = d + \mathrm{ad}(dF).$$

2.3. **A connection argument.** In this subsection we review a general result (Lemma 2.3) on connections and prove a basic theorem (Theorem 2.4) which describes left module of rank 1 over the Weyl algebra A. To somewhat ease notations, we will denote by $\mathbb{N}_p$ the set of non negative integers which is less than p. Namely,

$$\mathbb{N}_p = \{0, 1, 2, \dots, p-1\}.$$

Lemma 2.3. *Let k be a commutative ring of characteristic p. Let S be a commutative k-algebra. We assume that S contains elements $\{t_1, t_2, \dots, t_d\}$ such that $\Omega^1_{S/k}$ is freely generated by $\{dt_1, dt_2, \dots, dt_d\}$ over S. Let M be an S-module. Let*

$$\nabla : M \to \Omega^1_{S/k} \otimes_S M$$

be a connection on M. Then the following conditions are equivalent.

1. *M is generated by parallel sections.*
2. *The curvature and the p-curvature of ∇ are both zero.*
3. *$M = \sum_{J \in \mathbb{N}_p^d} t^J M^\nabla$.*

Proof. (1) $\implies$ (2): Curvatures and p-curvatures are tensors. Condition (1) implies that they vanish on generators on M. So they are zero.
(3) $\implies$ (1): trivial.
(2) $\implies$ (3): is a result of Taylor expansion formula. Namely, for any element $m \in M$, we have

$$m = \sum_{J \in \mathbb{N}_p^d} \frac{1}{J!} m_J t^J \tag{TE}$$

where the coefficients are computed in the following manner.

$$m_J = \left(\sum_{I \in \mathbb{N}_p^d} \frac{1}{I!} (-t)^I (\nabla_t)^{I+J} m \right) \in M^\nabla. \tag{TEC}$$

The uniqueness of such expansion follows in a similar manner as the uniqueness of the usual Taylor expansion of C^∞-functions.

□

Theorem 2.4. *Let W be a left A-module of rank 1. Then the following conditions are equivalent*

1. *W is a projective A-module.*
2. *There exists a left A-module W' such that $W \oplus W'$ is A-free.*
3. *W is R-projective.*
4. *W is R-locally free.*
5. *$W \otimes_R R^{(1/p)} \cong M_{p^n}(R^{(1/p)})$ as an $A \otimes_R R^{(1/p)} \cong M_{p^n}(R^{(1/p)})$-module.*
6. *$\oplus_{I \in \mathbb{N}_p^{2n}} T^I W \cong M_{p^n}(R^{(1/p)})$ as A-module.*
7. *There exists an A-linear connection ∇ on $M_{p^n}(R^{(1/p)})$ such that*
   (a) $\mathrm{curv}(\nabla) = 0$
   (b) $p\text{-}\mathrm{curv}(\nabla) = 0$
   (c) $W = M_{p^n}(R^{(1/p)})^\nabla$.
8. *$\oplus_{I \in \mathbb{N}_p^{2n}} \mu^I W \cong M_{p^n}(R^{(1/p)})$ as A-module.*

Proof. (1) $\iff$ (2): follows from a general result on projective modules.
(2) $\implies$ (3): A is a free R-module.
(3) $\implies$ (4): is also an elementary result of commutative algebra.
(4) $\implies$ (5): $W^{(1/p)} = W \otimes_R R^{(1/p)}$ is a left $A \otimes_R R^{(1/p)} \cong M_{p^n}(R^{(1/p)})$ module of rank 1. By using an argument on elementary matrices $\{e_{ij}\}$, we see that $W^{(1/p)}$ is isomorphic to a tensor product of the standard "vector representation" of $M_{p^n}(R^{(1/p)})$ and $e_{11}W^{(1/p)}$. Now, $e_{11}W^{(1/p)}$ is a direct summand of $R^{(1/p)}$-projective module $W^{(1/p)}$. So $e_{11}W^{(1/p)}$ is locally free. By a theorem of Quillen-Suslin (also known as "Serre conjecture"), we see that $e_{11}W^{(1/p)}$ is a free $R^{(1/p)}$-module of rank p^n. Thus $W^{(1/p)}$ is isomorphic to $M_{p^n}(R^{(1/p)})$ as an $A \otimes_R R^{(1/p)} \cong M_{p^n}(R^{(1/p)})$-module.
(5) $\implies$ (6): $R^{(1/p)} \cong \oplus_{I \in \mathbb{N}_p^{2n}} RT^I$.

(6) $\implies$ (2): $M_{p^n}(R^{(1/p)})$ is free A-module.
(6) $\iff$ (7): follows from the previous lemma.
(6) $\implies$ (8):

$$M_{p^n}(R^{(1/p)}) = \sum_I T^I W = \sum_I (\gamma - \mu)^I W \subset \sum_I \mu^I W$$

So a map

$$\oplus_{I \in \mathbb{N}_p^{2n}} W \ni (w_I) \mapsto \mu^I w_I \in M_{p^n}(R^{(1/p)})$$

is surjective. By comparing rank, we see that the map is generically injective. Since W has no torsion, we conclude that the map is surjective. (8) $\implies$ (2): trivial. □

2.4. **Endomorphisms of Weyl algebras.** For a given antisymmetric matrix h, let us define a symplectic form ω_h on $\mathbb{A}^{2n}$ as follows.

$$\omega_h = \sum_{i<j} h_{ij} dT_i dT_j.$$

Let $f : (\mathbb{A}^{2n}, \omega_h) \to (\mathbb{A}^{2n}, \omega_h)$ be a symplectic endomorphism of degree $d \leq \frac{p-1}{2}$. Then by using F defined in 2.2.2 we define a connection ∇^{gauge} on $M_{p^n}(R^{(1/p)})$ as follows

$$\nabla^{\text{gauge}} = d + \lambda(dF) - \varrho(d(f^*F)).$$

We define $W^{(f)}$ to be the set of parallel sections. That means,

$$W^{(f)} = M_{p^n}(R^{(1/p)})^{\nabla^{\text{gauge}}}.$$

Proposition 2.5. *Under the assumption as above, we have:*

1. *The curvature and the p-curvature of ∇^{gauge} are both equal to* 0.
2. ∇^{gauge} *is A_n-linear.*
3. $W^{(f)}$ *is projective A_n-module of rank* 1.
4. *If $W^{(f)}$ is trivial as an A_n-module, then f is liftable to an endomorphism of A_n.*

Proof. (1): See the computation in [13, Proposition 2.3].
(2):A_n is the set of parallel section with respect to

$$\nabla = d + \text{ad}(dF).$$

It is easy to see that ∇ and ∇^{gauge} are compatible. That means,

$$\nabla^{\text{gauge}}(am) = \nabla(a)m + a\nabla^{\text{gauge}}m$$

holds for any sections $a, m \in M_{p^n}(\mathcal{O}_X)$. Then we see immediately that the connection ∇^{gauge} is A_n-linear.
(3): is a consequence of Theorem 2.4.
(4): Let G be the generating section of $W^{(f)}$. Then by reversing the order of the arguments done in [13] we see that a correspondence

$$a \to Gf^*(\Phi(a))G^{-1}$$

defines an endomorphism of the Weyl algebra A_n. □

In view of the proposition above, we may ask:

Problem 2.6. [2] Given a symplectic endomorphism f with total degree $\leq \frac{p-1}{2}$, is $W^{(f)}$ always free?

2.5. **Definition of homotopy.** Let f be a symplectic endomorphism f of $(\mathbb{A}^{2n}, \omega)$ where the symplectic form is given by

$$\omega = \sum_{i,j} h_{ij} dT_i T_j$$

It is known (for example in [7],[4]) that f is "homotopic" to identity.

One may ask:

Problem 2.7. Let W be a A_n-module of rank 1 which is homotopic to identity. Is W always free?

For a definition of "homotopy" of A_n-modules, we employ the following.

Definition 2.8. Let us consider $A_n = k\langle \xi_1, \xi_2, \dots, \xi_n, \eta_1, \eta_2, \dots, \eta_n \rangle$ as a subalgebra of $A_{n+1} = k\langle \xi_1, \xi_2, \dots, \xi_n, \xi_{n+1}, \eta_1, \eta_2, \dots, \eta_n, \eta_{n+1} \rangle$. Let us put $t = \xi_{n+1}$. Then projective left A_n-modules I_0, I_1 are said to be **homotopic** if there exists an $A_n[t]$-module $\tilde{J}$ such that

(1) $(A_n[t]/(t)) \otimes_{A_n[t]} \tilde{J} \cong I_0$ as an A_n-module.
(2) $(A_n[t]/(t-1)) \otimes_{A_n[t]} \tilde{J} \cong I_1$ as an A_n-module.
(3) $A_{n+1} \otimes_{A_n[t]} \tilde{J}$ is an A_{n+1}-module of rank 1 which is locally free as a R_{n+1}-module.

The condition (3) above asserts that $\tilde{J}$ behaves nicely with respect to the additional variable $t = \xi_{n+1}$. We shall show in the next section that the A_n-module $W^{(f)}$ associated to any symplectic endomorphism f of degree $d \leq \frac{p-1}{2}$ is actually homotopic to the trivial A_n-module in the above sense. It turns out in section 6 that with our definition of homotopy, the answer to the second problem is No, even for the case where $n = 1$.

2.6. **Every $W^{(f)}$ is homotopic to identity (if $\deg(f)$ is small).** In this subsection we prove the following proposition.

Proposition 2.9. *Let f be a symplectic endomorphism f of $(\mathbb{A}^{2n}, \omega)$ where the symplectic form is given by*

$$\omega = \sum_{i,j} h_{ij} dT_i T_j.$$

We assume that the degree d of f is $d \leq \frac{p-1}{2}$. Then $W^{(f)}$ is homotopic to the trivial A_n-module.

As we have said it is known that f is "homotopic" to identity. Let us recall the idea. By composing an affine coordinate change we may assume $f(0) = 0$ and $f'(0) = id$. In other words, we assume that the Taylor expansion of f at the origin 0 is written as

$$f(x) = x + \text{higher order terms}.$$

Then we define

$$f_t(x) = f(tx)/t.$$

We may easily see that $f_t(x)$ is actually a polynomial in x, t and that f_t is symplectic whenever we specialize t to an element of k.

We now proceed and use the homotopy map f_t as a component of a symplectic endomorphism $\hat{f}$ of $\mathbb{A}^{2n} \times \mathbb{A}^2$ (by adding the homotopy variable t and another extra variable u). To do that, we need an easy version of "Poincaré's lemma":

Lemma 2.10. *Let p be an odd prime. Let k be a field of characteristic p. Let α be a regular 1-form on an affine space $\mathbb{A}^m = \operatorname{Spec}(k[T_1, T_2, T_3, \dots, T_m])$. We assume:*

(1) $d\alpha = 0$.

(2) $\alpha = \alpha_j dT_j$ *with* $\deg(\alpha_j) < p-1 \quad (\forall j)$.

Then there exists a regular function a *on* $\mathbb{A}^m$ *such that* $da = \alpha$.

Proof. The result is an immediate consequence of the exact sequence [6, 2.1.11] which states that a closed form is exact if and only if it belongs to the kernel of the Cartier operator. Indeed, the form α surely belongs to the kernel since it satisfies the condition (2) above. To make ourselves somewhat self-contained, let us re-prove our special case. For each $x \in \mathbb{A}^m$, let us consider a "line"

$$\ell_x : \mathbb{A}^1 \ni t \to tx \in \mathbb{A}^m$$

and define

$$a(x) = \int_0^1 \ell_x^* \alpha.$$

It is easy to see that the integral a exists and satisfies the required condition $da = \alpha$. □

Convention 2.11. Before stating the next lemma, we explain some notational conventions. We would like to consider an affine space $\mathbb{A}^{2n}$ and its product $\mathbb{A}^{2n} \times \mathbb{A}^2$ with a plane. The exterior derivative d on $\mathbb{A}^{2n} \times \mathbb{A}^2$ then breaks into a sum

$$d = d_1 + d_2$$

where the d_1 (respectively, d_2) is the exterior derivative with respect to the first $2n$ (respectively, last 2) variables. We have the following relations of the exterior derivatives

$$ds = d_1 s + d_2 s = d_1 s + s' dt + \frac{\partial s}{\partial u} du \qquad (s' \overset{\text{def}}{=} \frac{\partial s}{\partial t}.) \tag{2.6.1}$$

for any form s on $\mathbb{A}^{2n} \times \mathbb{A}^2$.

Lemma 2.12. *Let* p *be an odd prime. Let* $f_t : \mathbb{A}^{2n} \to \mathbb{A}^{2n}$ *be a regular family of symplectic maps such that* $\deg(f_t) \leq \frac{p-1}{2}$. *That means, we assume that there exists a polynomial map*

$$\tilde{f} : \mathbb{A}^{2n} \times \mathbb{A}^1 \to \mathbb{A}^{2n}$$

such that $f_t(x) = \tilde{f}(x,t)$ *holds and each of* f_t *preserves the symplectic form* ω. *Then there exists a polynomial* $a(x,t)$ *on* $\mathbb{A}^{2n} \times \mathbb{A}^1$ *such that a map* $\hat{f}$ *defined by*

$$\hat{f} : \mathbb{A}^{2n} \times \mathbb{A}^2 \ni f(x,t,u) \mapsto (f_t(x), t, u + a(x,t)) \in \mathbb{A}^{2n} \times \mathbb{A}^2 \tag{2.6.2}$$

is symplectic with respect to the symplectic form $\hat{\omega} = \omega + dtdu$.

Proof. We first consider $\mathbb{A}^{2n}$ as the base space and consider differentials and forms on it. Let us define an 1-form

$$\rho = \sum_{i<j} h_{ij} T_i dT_j.$$

Then we have

$$\omega = d\rho.$$

Since each of f_t preserves the symplectic form, we have

$$d\rho = \omega = f_t^* \omega = f_t^* d\rho = df_t^* \rho.$$

By differentiating by t, we have

$$d((f_t^* \rho)') = 0. \tag{2.6.3}$$

Then by the Poincaré's lemma (Lemma 2.10 above), we deduce that there exists a function $a = a(x,t)$ such that

$$da = -(f_t^* \rho)'$$

holds.

Let us now turn our attention to the space $\mathbb{A}^{2n} \times \mathbb{A}^2$ and employ it as the base space instead of $\mathbb{A}^{2n}$. Let us define a map $\hat{f}$ by using the function a obtained as above and the equation 2.6.2 given in the statement of the lemma. By using the projection $\pi_1 : \mathbb{A}^{2n} \times \mathbb{A}^2 \to \mathbb{A}^{2n}$ to the first $2n$ variables, we may reinterpret the equation above as an equation

$$d_1 a = -(\hat{f}^* \pi_1^* \rho)' \tag{2.6.4}$$

of 1-forms on $\mathbb{A}^{2n} \times \mathbb{A}^2$. We then have

$$\begin{aligned}
\hat{f}^*\hat{\omega} &= \hat{f}^*(dtdu + \pi_1^* d\rho) \\
&= dt(du + da) + \hat{f}^*\pi_1^* d\rho && \text{(the definition of } \hat{f}) \\
&= dt(du + da) + d(\hat{f}^*\pi_1^*\rho) && \text{(functoriality of } d) \\
&= dt(du + da) + (\hat{f}^*\pi_1^*\rho)' dt + d_1(\hat{f}^*\pi_1^*\rho) && \text{(by 2.6.1)} \\
&= dt(du + d_1 a + a' dt) - dt(d_1 a) + d_1(\hat{f}^*\pi_1^*\rho) && \text{(by 2.6.4)} \\
&= dtdu + d_1\omega && (f_t \text{ preserves } \omega) \\
&= \hat{\omega}
\end{aligned}$$

That means, the map $\hat{f}$ is symplectic.

□

Let us now apply the argument carried out in subsection 2.4 to the extended symplectic map $\hat{f}$. Aside from the existing variables (matrices) $\{\mu_j\}_{j=1}^{2n}$ which appears in the equation 2.2.1 in $2n$ variables, we introduce extra variables ν_0, μ_0 which commute with other existing variables and satisfy

$$[\nu_0, \mu_0] = 1, \quad \nu_0^p = 0, \quad \mu_0^p = 0.$$

We put

$$\hat{F} = \pi_1^* F - t\nu_0 + u\mu_0$$

where F is given by the equation 2.2.2 in the original $2n$ variables. Namely,

$$F = \sum_{i,j=1}^{2n} \bar{h}_{ij}\mu_j T_i.$$

Let us denote by $\nabla^{(\hat{f})}$ the connection ∇^{gauge} associated to the extended symplectic map $\hat{f}$. In concrete terms, we have

$$\begin{aligned}
\nabla^{(\hat{f})} &= d + \lambda(d\hat{F}) - \varrho(d\hat{f}^*\hat{F}) \\
&= d + \lambda(\pi_1^* dF - \nu_0 dt + \mu_0 du) - \varrho(\hat{f}^*\pi_1^* dF - \nu_0 dt + \mu_0(du + da_0)) \\
&= d_1 + d_t + d_u + \lambda(d_1(\pi_1^* F) - \nu_0 dt + \mu_0 du) \\
&\quad - \varrho(d_1(\hat{f}^*\pi_1^* F) + (\hat{f}^*\pi_1^* F)' dt - \nu_0 dt + \mu_0(du + da_0)).
\end{aligned}$$

Let us consider the space

$$J^{(f)} = k[T, t, u, \mu, \mu_0, \nu_0]^{\nabla^{(\hat{f})}}$$

of parallel elements with respect to our connection $\nabla^{(\hat{f})}$. (To simplify the notation, we denote by μ the $2n$ variables $\mu_1, \mu_2, \ldots \mu_{2n}$.) Using it we construct an module $\tilde{J}^{(f)}$ as in Definition 2.8. To somewhat simplify the calculation, we use a (truncated) exponential function

$$\mathrm{ex}_p(x) = \sum_{j=0}^{p-1} \frac{1}{j!} x^j$$

and consider the following conjugation of $\nabla^{(\hat{f})}$.

$$\begin{aligned}\nabla^{[\hat{f}]} =& \exp_p(-t\,\mathrm{ad}\,\nu_0)\nabla^{(\hat{f})}\exp_p(t\,\mathrm{ad}\,\nu_0)\\ =& d_1 + \lambda(d_1\pi_1^*F) - \varrho(d_1(\hat{f}^*\pi_1^*F) + \epsilon_0(d_1a_0)\\ &+ \partial_t - \varrho((\hat{f}^*\pi_1^*F)' + \epsilon_0 a_0')dt\\ &+ \partial_u + \mathrm{ad}(\epsilon_0)du\end{aligned}$$

where we put

$$\epsilon_0 = \mu_0 - t - \mu_0^{p-1}t^p.$$

The variable ϵ_0 satisfies the following relations.

$$\epsilon_0^p = 0, \qquad \frac{\partial}{\partial t}\epsilon_0 = -1.$$

The module $J^{(f)}$ is then linearly isomorphic to the set $J^{[f]}$ of parallel elements with respect to $\nabla^{[\hat{f}]}$ via the multiplication by $\exp_p(t\,\mathrm{ad}\,\nu_0)$. In what follows, we will work on this conjugated space. Let us decompose the conjugated connection $\nabla^{[\hat{f}]}$ with respect to the decomposition $\mathbb{A}^{2n+2} = \mathbb{A}^{2n}\times\mathbb{A}^1\times\mathbb{A}^1$.

$$\begin{aligned}\nabla^{[\hat{f}]}_{d/dT} =& d_1 + \lambda(d_1\pi_1^*F) - \varrho(d_1(\hat{f}^*\pi_1^*F) + \epsilon_0(d_1a_0)\\ \nabla^{[\hat{f}]}_{d/dt} =& \partial_t - \varrho((\hat{f}^*\pi_1^*F)' + \epsilon_0 a_0')dt\\ \nabla^{[\hat{f}]}_{d/du} =& \partial_u + \mathrm{ad}(\epsilon_0)du\end{aligned}$$

Since the curvature of $\nabla^{(\hat{f})}$ is equal to zero, the curvature of $\nabla^{[\hat{f}]}$ is also equal to zero. We may thus consider above three connections separately. Namely, we have

$$J^{[f]} = k[T,t,u,\mu,\mu_0,\nu_0]^{\nabla^{[\hat{f}]}} = \left((k[T,t,u,\mu,\mu_0,\nu_0]^{\nabla^{[\hat{f}]}_{d/du}})^{\nabla^{[\hat{f}]}_{d/dt}}\right)^{\nabla^{[\hat{f}]}_{d/dT}}.$$

We may easily integrate the equation of parallelism with respect to u and see that

$$k[T,t,u,\mu,\mu_0,\nu_0]^{\nabla^{[\hat{f}]}} = k[u+\nu_0]\otimes_k \tilde{J}^{[f]}$$

holds where we have put

$$\tilde{J}^{[f]} = k[T,t,\mu,\epsilon_0]^{\nabla^{[\hat{f}]}} = \{x\in k[T,t,\mu,\epsilon_0]; \nabla^{[\hat{f}]}_{d/dT}(x) = 0, \nabla^{[\hat{f}]}_{d/dt}(x) = 0\}.$$

We show that the module $\tilde{J}^{[f]}$ plays (after multiplied back with $\exp_p(-t\,\mathrm{ad}\,\nu_0)$) a role of $\tilde{J}$ in the Definition 2.8. Using Lemma 2.3 we may immediately verify the condition 3 of the definition. To verify the condition 1,2 of the definition, let us integrate with respect to t. We use the equation (TEC) which appears in the Taylor expansion in the proof of Lemma 2.3 and consider the following linear operator.

$$Lx = \sum_{k=0}^{p-1}\frac{1}{k!}\epsilon_0^k(\nabla^{[\hat{f}]}_{\frac{\partial}{\partial t}})^k x.$$

We introduce a new variable

$$\tau = t + \epsilon_0$$

which commutes with every element of $k[T,t,\mu,\epsilon_0]$ and with the derivation $\frac{\partial}{\partial t}$. Using the nilpotency of ϵ_0 we see that the linear operator L gives an k-linear isomorphism

$$k[T,\tau,\mu] \cong k[T,t,\mu,\epsilon_0]^{\nabla^{[\hat{f}]}_{d/dt}}$$

whose inverse operator being given by the (restriction of) specialization $\epsilon_0 \to 0$. As we have mentioned, the curvature of $\nabla^{[f]}$ is zero so that L commutes with $\nabla^{[\hat{f}]}_{d/dT}$. We thus see that we have

$$\tilde{J}^{[f]} = k[T, t, \mu, \epsilon_0]^{\nabla^{[\hat{f}]}} \cong k[T, \tau, \mu]^{\left(\nabla^{[\hat{f}]}_{d/dT}|_{\epsilon_0=0}\right)}.$$

The right hand side draws back to $W^{(f)}$ (respectively, the trivial A_n-module) when restricted to $\tau = 1$ (respectively, $\tau = 0$.) We have now verified that the condition 1,2 of Definition 2.8 is satisfied and our proposition is proved.

3. Definition of reflexivity

In this paper, the word "reflexive" is used in two ways. One is A-reflexivity which is defined as follows.

Definition 3.1. (1) For any left A-module W, we define its dual $W^{\triangleright}$ as a right A-module defined by

$$W^{\triangleright} = \mathrm{Hom}_A(W, A)$$

(2) Similarly, for any right A-module V, we define its dual $V^{\triangleleft}$ as a left A-module defined by

$$V^{\triangleleft} = \mathrm{Hom}_{\text{-}A}(V, A).$$

Definition 3.2. An A-module W is called **A-reflexive** if the canonical homomorphism

$$\mathrm{eval} : W \to (W^{\triangleright})^{\triangleleft}.$$

is an isomorphism.

We would like to interpret A-reflexivity in a geometric way. Here comes the reflexivity as in the usual sense in algebraic geometry like in [5]. Namely, we would like to interpret A-reflexivity of an A-module W by a reflexivity of $\mathcal{O}_X$-module for the corresponding sheaf W_X. (See the convention 2.2 for the meaning of W_X.)

Definition 3.3. [1, p.128],[5, p.126] A coherent sheaf $\mathcal{F}$ on X is **normal** if for every open set $U \subset X$ and every closed subset $Y \subset U$ of codimension ≥ 2, the restriction map $\mathcal{F}(U) \to \mathcal{F}(U \setminus Y)$ is bijective.

Proposition 3.4. [5, Proposition 1.6]. *An sheaf on a normal integral scheme is reflexive if and only if it is torsion free and normal*

As a result of arguments on connections, we see that locally free R-module is actually A-locally free, as the following lemma states.

Lemma 3.5. *Let W_U be a left A_X module on an open subset U of X. We assume that W_U is a $\mathcal{O}_U$-free module. Then W_U is locally isomorphic to $A_U^{\oplus r}$ as an A_U-module.*

Proof. By a general theory on full matrix algebras, we see that $W_U \otimes_R R^{(1/p)}$ is locally isomorphic to $M_{p^n}(R^{(1/p)})^{\oplus r}$. Let us equip $M_{p^n}(R^{(1/p)})^{\oplus r}$ with a connection ∇^W compatible with the isomorphism. Then the curvatures and p-curvature of ∇^W is equal to 0. Thus by using "Taylor's formula"(TE), for each closed point P of U we may easily construct a set of local sections $B = \{w_1, w_2, \ldots, w_r\}$ of W_U which equals to an $A|_P (\cong M_{p^n}(k_P))$-basis of the fiber $W|_P$ of W at P. Then B is an A-basis of W in a neighborhood of P. Thus W_U is locally isomorphic to A_U as required.

$\square$

Then we have the following criteria for reflexivity.

Proposition 3.6. *The following conditions are equivalent:*

1. *W is A-reflexive.*
2. *$W \cong M^{\triangleleft}$ for a right A-module M.*
3. *W is A-torsion free and W_X is normal on X.*
4. *W is A-torsion free and R-reflexive.*

Proof. (1) $\Longrightarrow$ (2): Put $M = W^{\triangleright}$.
(2) $\Longrightarrow$ (3): Since we know that A has no zero-divisors, we see that $M^{\triangleleft}$ has no A-torsion. It is also easy to see that $M^{\triangleleft}$ is normal.
(3) $\Longleftrightarrow$ (4): is a consequence of Proposition 3.4.
(4) $\Longrightarrow$ (1): There exists a closed subset F of codimension ≥ 2 such that W_X is $\mathcal{O}_X$-locally free on $U = X \setminus F$. In view of Lemma 3.5, we see that W_X is A_X-locally free on U. Thus the sheaf homomorphism

$$\mathrm{eval}_X : W_X \to \mathrm{Hom}_{\text{-}A_X}(\mathrm{Hom}_{A_X}(W_X, A_X), A_X) = ((W^{\triangleright})^{\triangleleft})_X$$

induced by eval is an isomorphism when restricted to U. Since both W and $(W^{\triangleright})^{\triangleleft}$ are R-reflexive, this implies that eval gives an isomorphism on the whole of X. □

Corollary 3.7. *Intersections of reflexive modules are reflexive. In particular, for any elements $\alpha_1, \alpha_2, \ldots, \alpha_s \in A$, the intersection*

$$A\alpha_1 \cap A\alpha_2 \cap \cdots \cap A\alpha_s$$

is reflexive.

A result of Stafford [10] shows that any reflexive A-modules of rank one are obtained in the form of the corollary above with $s = 2$. We will see later in Theorem 5.4 an alternate proof of the fact using our theory. (In fact, the result of Stafford is valid in a much general context so that it may deal with any prime Goldie ring A in place of the Weyl algebra.)

4. Norms

4.1. **Norm on the Weyl algebra A.** We define a norm map $N_A : A \to R^{(1/p)}$ by the following diagram:

$$N_A : A \overset{\otimes 1}{\to} A \otimes_R R^{(1/p)} \overset{\Phi_0}{\cong} M_{p^n}(R^{(1/p)}) \overset{\det}{\to} R^{(1/p)}.$$

Proposition 4.1. *The norm N_A satisfies the following conditions:*

1. *$N_A(x) \in R$ for any $x \in A$.*
2. *$N_A(xy) = N_A(x)N_A(y)$ for any $x, y \in A$.*

4.2. **Norm on a reflexive A-module W of rank one.**

Lemma 4.2. *Let W be a reflexive A-module of rank one. Then there exists an injective A-module homomorphism*

$$W \to A.$$

Thus W is isomorphic to a left ideal of A.

Proof. We take a non zero element φ of $W^{\triangleright}$. φ is generically an isomorphism. Since W is torsion free, the kernel of φ should be equal to zero. □

Proposition 4.3. *For any reflexive left A-module W of rank one, there exists a map*

$$N_W : W \to R$$

which satisfies:

1. *$N_W(ax) = N_A(a)N_W(x)$ for all $a \in A$ and for all $x \in W$.*

(2) *For any open set $U \subset \mathbb{A}^{2n}$, $x \in W_X(U)$ is a generating section of W_U if and only if $N_W(x)$ is invertible on U.*

Furthermore, the map is unique up to a multiplication by a constant in $k^\times$.

Proof. We may assume that W is equal to a left ideal J of A. Let us put

$$c_J = \gcd\{N_A(x); x \in J\}. \tag{4.2.1}$$

The greatest common divisor c_J exists since the k-algebra R is a unique factorization domain.

Let us then put

$$N_J(x) = N_A(x)/c_J.$$

The property (1) is then trivially satisfied. To prove the property (2), let us first assume that W_U is free. Then we have $W_U \otimes_{\mathcal{O}_U} \mathcal{O}^{(1/p)} \cong M_{p^n}(\mathcal{O}^{(1/p)})$ as an $A_U \otimes_{\mathcal{O}_U} \mathcal{O}^{(1/p)} \cong M_{p^n}(\mathcal{O}^{(1/p)})$-module. We see immediately that N_J coincides with a multiple of the determinant map det by an invertible element of $\mathcal{O}_X(U)$. Thus N_J satisfies the property (2) in this case. For general case, we may use the fact that there exists an closed subset F of codimension ≥ 2 in U such that $W_{U\setminus F}$ is locally free. □

Definition 4.4. For any reflexive module W of rank one, We call the map N_W as in the proposition the **norm map** of W.

Lemma 4.5. *For any $y \in W$, we have $W \subset A \cdot y N_J(y)^{-1}$.*

Proof. There exists a closed subset F of codimension ≥ 2 in X such that W_U is locally free on $U = X \setminus F$. That means, there exists an open covering $\{V_j\}$ of U such that W_{V_j} is free. Then on each V_j we have

$$W_{V_j} \otimes_{\mathcal{O}} \mathcal{O}^{(1/p)} \cong M_{p^n}(\mathcal{O}^{(1/p)})$$

as an $A \otimes_{\mathcal{O}} \mathcal{O}^{(1/p)} \cong M_{p^n}(\mathcal{O}^{(1/p)})$-module. With this identification, we see that for any $x, y \in W$, $xy^{-1}N_J(y)$ defines a section a_j of $\Gamma(V_j, M_{p^n}(\mathcal{O}^{(1/p)}))$ which is parallel with respect to ∇^A. In other words, there exists a unique section $a_j \in \Gamma(V_j, A_X)$ such that

$$x = a_j y N_J(y)^{-1}.$$

By the uniqueness, these $\{a_j\}$ patch together to define a section $a \in \Gamma(X \setminus F, A_X)$. Since the codimension of F is ≥ 2, a extends to the whole of X, that means, it is actually an element of A and it satisfies $x = ayN_J(y)^{-1}$ as required. □

5. Structure of reflexive modules of rank one.

Using the norm defined in the preceding section, we show that a reflexive A-module of rank one is an intersection of a examine a structure of reflexive modules of rank one. Although the fact is known to be true in a much more general context ([10, Corollary 3.9]), we give an alternative proof for the sake of completeness.

Definition 5.1. For an A-module W of rank one, a subset $\{w_1, w_2, \ldots, w_s\}$ of W is called **weakly generating** if

$$\gcd\{N_W(w_1), N_W(w_2), \ldots, N_W(w_s)\} = 1.$$

Proposition 5.2. *Let J be a reflexive left ideal of A. Then a subset $\{w_1, w_2, \ldots, w_s\}$ of J is weakly generating if and only if*

$$J = \bigcap_{j=1}^{s} A w_j N_J(w_j)^{-1}.$$

Proof. Let us denote by $\overline{J}$ the right hand side. Using Lemma 4.5, we may easily see that $J \subset \overline{J}$ holds. To prove the other inclusion, let us take an element x of $\overline{J}$. For each j, let us denote by U_j the open set defined by

$$U_j = \{N_J(w_j) \neq 0\}.$$

Then for each j, the element x is an element of $Aw_j N_J(w_j)^{-1}$ and so it is an regular element of W on U_j. Thus x is regular on $V = \cup_{j=1}^s U_j$. Since $\{w_1, w_2, \dots, w_s\}$ is weakly generating, $\mathbb{A}^{2n} \setminus V$ is of codimension greater than or equal to 2. By the normality of J, we see that $x \in J$ as required. □

In a paper of J. T. Stafford[9], it is shown that any left ideal of $A_n(k)$ is generated by two elements when the characteristic of the base field k is equal to 0. When the characteristic of the base field k is non-zero, then the corresponding statement is trivially false. (To see this, we consider ideals generated by elements of R such as $J = \sum_{j=1}^{2n} A_n \gamma_j^p$.) We give a much weaker, but easier, proposition.

Proposition 5.3. *For any reflexive A-module W of rank 1, there exists a weakly generating sections $\{w_1, w_2\}$.*

Proof. In short, it is a result of "independence of valuations." Let $w_0 \in W$ be an non zero element. Let us decompose the zero divisor of $N_W(w_0)$ into irreducible components.

$$\{N_W(w_0) = 0\} = \cup_{j=1}^s D_j.$$

For each j, let us take $w_j \in W$ such that $N_W(w_j)|_{D_j} \neq 0$. (Such an element exists since $\cap_{w \in W}\{N_W(w) = 0\} = \emptyset$.) Let us denote by $z_j \in R$ the defining function of D_j. Then we put

$$w_\infty = \sum_{j=1}^s (\prod_{k \neq j} z_k) w_j.$$

Then we see immediately that $\{(N_W(w_0) = 0\} \cap \{N_W(w_\infty) = 0\}$ is of codimension greater than or equal to 2 in $\operatorname{Spec} R$.

□

Theorem 5.4. *A left A-module W of rank one is A-reflexive if and only if W is isomorphic to a left ideal $J = A\alpha \cap A\beta$ for some $\alpha, \beta \in A$. Furthermore, if that is the case, we may choose J, α, β in such a way that $\operatorname{lcm}(N_A(\alpha), N_A(\beta)) = c_J$. (See equation c_J for the definition of c_J.)*

Proof. We already know in Lemma 3.7 that $J = A\alpha \cap A\beta$ is reflexive for any element $\alpha, \beta \in A$. So any module W isomorphic to the ideal J obtained in this way is reflexive. Conversely, assume we have a reflexive A-module W of rank 1. We may assume that W is equal to a left ideal I of A. We may obtain by Proposition 5.3 an weakly generating sections $\{w_1, w_2\}$ of I. Then by Proposition 5.2 we see that I may be written as a intersection

$$I = Aw_1 N_I(w_1)^{-1} \cap Aw_2 N_I(w_2)^{-1}.$$

By multiplying by $N_I(w_1) N_I(w_2)$, I is isomorphic to

$$J = Aw_1 N_I(w_2) \cap Aw_2 N_I(w_1).$$

We see by a direct computation that $J, \alpha = w_1 N_I(w_2), \beta = w_2 N_I(w_1)$ satisfy the condition $\operatorname{lcm}(N_A(\alpha), N_A(\beta)) = c_J$ as required.

□

With the help of the theorem above, we may easily construct examples of reflexive left A modules of rank one. All we need is to take arbitrary pair of elements α, β in A and consider the intersection $A\alpha \cap A\beta$. We note, however, that there are some cases where the condition $\operatorname{lcm}(N(\alpha), N(\beta)) \neq c_J$ is not met for $J = A\alpha \cap A\beta$.

Example 5.5. Let $n = 1$, $\alpha = \xi_1\eta_1$, $\beta = \xi_1\eta_1 - 1$. Then $N(\alpha) = N(\beta) = \xi_1^p\eta_1^p$ (see the next lemma for a computation). It is easy to see that we have in this case

$$J = A\alpha \cap A\beta = A\alpha\beta$$

so that we have

$$c_J = N(\alpha)N(\beta) \neq \mathrm{lcm}(N(\alpha), N(\beta)).$$

Lemma 5.6. *We have the following identity in $A = A_1(k)$. (For simplicity's sake, we we put $\xi = \xi_1, \eta = \eta_1$.)*

(1) $\xi^t\eta^t = (\xi\eta)(\xi\eta - 1)(\xi\eta - 1)\dots(\xi\eta - (t-1))$
(2) $(\xi\eta)^p - \xi\eta = \xi^p\eta^p$.
(3) $N(\xi\eta) = \xi^p\eta^p$.

Proof. (1) Let us put $\theta = \xi\eta$. Then we have $\eta\theta = (\theta + 1)\eta$. It is easy to see that $\xi^i\eta^i$ is a polynomial $f_i(\theta)$ in θ, and that the polynomials $\{f_i\}$ satisfy the following inductive formula.

$$f_{i+1}(\theta) = \xi f_i(\theta)\eta = \xi\eta f_i(\theta - 1) = \theta f_i(\theta - 1)$$

The equation follows easily from this.
(2) This is a special case of (1). We note that the relation (2) gives the minimal polynomial of $\theta = \xi\eta$ over k. We may thus easily see that (3) holds.

□

5.1. A note on R-locally free ideals of A.

Proposition 5.7. *Let us assume we are given an ideal $J = A\alpha \cap A\beta$ of A with generators α, β such that $\mathrm{lcm}(N_A(\alpha), N_A(\beta)) = c_J$. Then*

(1) *J is R-locally free if and only if there exists an element $x_J \in M_{p^n}(R^{(1/p)})\alpha \cap M_{p^n}(R^{(1/p)})\beta$ such that*
$$\det(x_J) = c_J$$
holds.
(2) *J is A-free if and only if we may choose x_J above as an element of A.*

Proof. (1): J is R-locally free if and only if $J \otimes_R R^{(1/p)}$ is locally free over $R^{(1/p)}$. In that case, by using the "Serre conjecture" and the matrix arguments, we see that $J \otimes_R R^{(1/p)}$ is isomorphic to $M_{p^n}(R^{(1/p)})$ as an $A \otimes_R R^{(1/p)} \cong M_{p^n}(R^{(1/p)})$-module. Thus J is R-locally free if and only if there exists an element $x_J \in M_{p^n}(R^{(1/p)})$ such that

$$M_{p^n}(R^{(1/p)})\alpha \cap M_{p^n}(R^{(1/p)})\beta = M_{p^n}(R^{(1/p)})x_J$$

holds.

Now, assume J is R-locally free and take an element x_J as above. From the fact that x_J is an element of the intersection

$$M_{p^n}(R^{(1/p)})\alpha \cap M_{p^n}(R^{(1/p)})\beta,$$

we see immediately that $\det(x_J)$ is a multiple of $\mathrm{lcm}(N(\alpha), N(\beta))$. On the other hand, from the definition of c_J, we see that c_J is a multiple of $\det(x_J)$. Since we assumed $c_J = \mathrm{lcm}(N(\alpha), N(\beta))$, we see that $\det(x_J)$ is a constant multiple of c_J.

Conversely, let us assume that such x_J exists. Then we have an inclusion

$$M_{p^n}(R^{(1/p)})x_J \subset M_{p^n}(R^{(1/p)})\alpha \cap M_{p^n}(R^{(1/p)})\beta.$$

The both hand sides are $R^{(1/p)}$-reflexive and the inclusion gives an isomorphism on an open set U of X such that $X \setminus U$ is of codimension ≥ 2. Thus we actually see that the inclusion is an equation.

(2): follows easily from the above argument and Proposition 4.3.

□

5.1.1. *connection* ∇^J. Using the element x_J as above, we may introduce a "defining derivation" ∇^J.

Lemma 5.8. *Let J be a projective left A-module of rank 1. There is a unique connection on $M_{p^n}(R^{(1/p)}) \cong M_{p^n}(R^{(1/p)})x_J = J \otimes_R R^{(1/p)}$ such that*

(1) *Each element of J is parallel with respect to ∇^J.*
(2) *∇^J is compatible with the action of $A \otimes_R R^{(1/p)} \cong M_{p^n}(R^{(1/p)})$.*

Proof. We put

$$\nabla^J(x) = \nabla^A(xx_J)x_J^{-1} = \nabla^A(x) + x \cdot \nabla^A(x_J)x_J^{-1}.$$

(There is of course several ways to define ∇^J above.) □

We may describe the moduli space of R-free(=projective) left A-module of rank one in a good old "connection modulo gauge group" style:

Definition 5.9. Let us temporarily say that a connection ∇^J on $M_{p^n}(\mathcal{O}^{(1/p)})$ is left compatible with ∇^A if

$$\nabla^J(xy) = \nabla^A(x)y + x\nabla^J(y)$$

holds for any $x, y \in M_{p^n}(\mathcal{O}^{(1/p)})$. Let us denote by $\mathcal{A}$ the set of all left compatible connections.

Theorem 5.10. *Let us put $K = Q(R)$, the quotient field of R. The projective left A modules of rank one are parametrized by*

$$\begin{aligned} &\mathrm{GL}_{p^n}(R^{(1/p)})\backslash\mathcal{A} \\ \cong\, &\mathrm{GL}_{p^n}(R^{(1/p)})\backslash\{x \in M_{p^n}(K); \nabla^A(x)x^{-1} \text{ is regular}\}/A \otimes_R K. \end{aligned}$$

6. An example.

We give an example of a reflexive left ideal of a Weyl algebra $A_2 = k\langle \xi_1, \xi_2, \eta_1, \eta_2 \rangle$. For simplicity, we put $\xi = \xi_1$, $\eta = \eta_1$, $t = \xi_2$.

Let us put

$$J = A_2(1 + t\xi) \cap A_2\eta.$$

This section is devoted to give some analysis of the example. First we give some definition to make our arguments easier.

Definition 6.1.

$$a_1 = (1 + t\xi) \tag{6.0.1}$$

$$a_3 = a_1\eta + 2t \tag{6.0.2}$$

$$b_1 = \eta + t(\xi\eta - 1) = a_1\eta - t \tag{6.0.3}$$

$$b_2 = \eta^2 \tag{6.0.4}$$

It is easy to see that following equation holds.

$$\eta a_1 = a_1\eta + t \tag{6.0.5}$$

Let us give a set of generators of J.

Proposition 6.2. *The ideal J is isomorphic to*

$$J_0 = A_2 b_1 + A_2 b_2 (= A_2 \cdot (\eta + t(\xi\eta - 1)) + A_2\eta^2)$$

as an A_2-module. More precisely, we have

$$J = J_0 a_1.$$

Proof. We want to find see when an element x of A_2 satisfies the condition $x(1+t\xi) \in A_2\eta$. Any element x of A_2 may be written as

$$x = x_2\eta^2 + x_1\eta + x_0$$

where

$$x_2 \in A_2, \quad x_1, x_0 \in k\langle \xi, t, \eta_2 \rangle.$$

Then

$$\begin{aligned}
& x \cdot a_1 \in A_2\eta \\
\iff & (x_1\eta + x_0) \cdot a_1 \in A_2\eta \\
\iff & x_1 t + x_0 a_1 \in A_2\eta \qquad \text{(by 6.0.5)} \\
\iff & x_1 t + x_0 a_1 = 0 \\
\iff & x_1 t + x_0 \cdot (1+t\xi) = 0 \\
\iff & x_1 t + x_0 \cdot (1+t\xi) = 0, \qquad x_0 = y_0 t \quad (\exists y_0 \in k\langle \xi, t, \eta_2 \rangle) \\
\iff & x_1 = -y_0 \cdot (1+t\xi), \qquad x_0 = y_0 t \quad (\exists y_0 \in k\langle \xi, t, \eta_2 \rangle) \\
\iff & x_1\eta + x_0 \in A_2 b_1
\end{aligned}$$

□

In the course of the proof above, we obtain the following equations which may be useful in dealing with the ideals J_0, J.

$$b_1 a_1 = a_1^2 \eta \tag{6.0.6}$$

$$b_2 a_1 = a_3 \eta \tag{6.0.7}$$

6.0.2. *Digression.* Since any reflexive sheaf which is reflexive over a normal variety of dimension ≤ 2 is locally free, by imitating the proof of the proposition above, we obtain the following proposition.

Proposition 6.3. *$A_1\xi \cap A_1\eta^{n+1}$ is projective A_1-module of rank one. It is isomorphic to $A_1(\xi\eta - n) + A_1\eta^{n+1}$.*

It should be noted that the ideals of this type are frequently studied from the earliest stage in the study of Weyl algebras.

6.1. **Projectivity of J_0.** Let us prove that our J_0 is R-locally free (that means, A-projective).

Proposition 6.4. *There exists an isomorphism*

$$J_0 \oplus A_2 \cong A_2 \oplus A_2$$

of left A_2-modules.

Proof. We have an exact sequence

$$0 \longrightarrow A_2 \xrightarrow{\cdot(-\eta,(1+t\xi))} A_2 \oplus A_2 \xrightarrow{\cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}} J_0 \longrightarrow 0.$$

Indeed, let us call the map $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ by ϕ. By the definition of J_0, ϕ is surely seen to be surjective. The kernel of ϕ is equal to

$$J_{\mathrm{Ker}} = A_2 b_1 \cap A_2 b_2.$$

We note that an equation

$$x_{J_{\mathrm{Ker}}} = \eta b_1 = a_1 b_2 \in J_{\mathrm{Ker}}$$

holds. By using Proposition 5.7, we see that the ideal J_{Ker} is trivial and generated by $x_{J_{\mathrm{Ker}}}$.

It is shown by a direct calculation that the splitting of the first arrow of the exact sequence above is given by

$$A_2 \oplus A_2 \xrightarrow{\cdot \begin{pmatrix} \xi a_1 \\ \xi\eta + 2 \end{pmatrix}} A_2.$$

□

6.2. **A homotopy between trivial and non trivial A-modules.** We put

$$\tilde{J} = A_1[t] \cdot b_1 + A_1[t] \cdot b_2.$$

Then it is an $A_1[t]$-module such that

$$I_0 = A_1 \cdot \eta + A_1 \cdot \eta^2 = A_1 \cdot \eta \cong (A_1[t]/t) \otimes_{A_1[t]} \tilde{J},$$

and

$$I_1 = A_1 \cdot ((\xi+1)\eta - 1) + A_1 \cdot \eta^2. \cong (A_1[t]/(t-1)) \otimes_{A_1[t]} \tilde{J}$$

holds. We also note that $J_0 \cong A_2 \otimes_{A_1[t]} J$ is stably free (hence is projective) by Proposition 6.4. Thus the A_1-modules I_0 and the I_1 are homotopic in the sense of Definition 2.8.

As the reader may see, I_0 is trivial A_1-module generated by η. On the other hand, I_1 is not trivial. (To see this, assume I_1 is trivial. Then I_1 should be generated by an element x in $A = A_1$ with norm $N_A(x) = \eta^p$. By looking at principal term, we see immediately that such x should equal to η. But by considering a representation

$$\Phi(\xi) = X, \quad \Phi(\eta) = \partial/\partial X$$

of A on $k[X]$, we may see that $\eta \notin I_1$, since we have

$$\Phi(a).(X+1) = 0$$

for any $a \in I_1$, where as $\Phi(\eta).(X+1) = 1 \neq 0$.)

References

[1] W. Barth, *Some properties of stable rank-2 vector bundles on $\mathbb{P}_n$*, Math. Ann. **226** (1977), 125–150.

[2] A. Belov-Kanel and M. Kontsevich, *Automorphisms of the Weyl algebra*, Lett. Math. Phys **74** (2005), no. 2, 181–199.

[3] Y. Berest and O. Chalykh, *A_∞-modules and Calogero-Moser spaces*, J. Reine Angew. Math. **607** (2007), 69–112.

[4] A. Essen, *Polynomial automorphisms and the Jacobian conjecture*, Birkhäuser, 2000.

[5] R. Hartshorne, *Stable reflexive sheaves*, Math. Ann. **254** (1980), 121–176.

[6] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Sci. Éc. Norm. Sup. (1979), 501–661.

[7] Vik. S. Kulikov, *The Jacobian conjecture and nilpotent mappings*, J. Math. Sci. **106** (2001), no. 5, 3312–3319.

[8] J. T. Stafford, *Weyl algebras are stably free*, J.Algebra **48** (1977), no. 2, 297–304.

[9] ———, *Module structure of Weyl algebras*, J. London Math. Soc.(2) **18** (1978), no. 3, 429–442.

[10] ———, *Modules over prime Krull rings*, J. Algebra **95** (1985), no. 2, 332–342.

[11] Y. Tsuchimoto, *Preliminaries on Dixmier conjecture*, Mem. Fac. Sci. Kochi Univ. Ser. A Math. **24** (2003), 43–59.

[12] ———, *Endomorphisms of Weyl algebra and p-curvatures*, Osaka J. Math. **42** (2005), no. 2, 435–452.

[13] ———, *Equations in p-curvature and intertwiners*, Osaka J. Math. **45** (2008), no. 3, 737–746.

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