No.20: Supplement.

LEMMA 20.1. *Let A be a commutative ring. Let n be a positive integer. Then:*

(1) *$V_n(\mathcal{W}_1(A))$ is an ideal of $\mathcal{W}_1(A)$.*
(2) *If n is invertible in A, then*

$$e_n \cdot \mathcal{W}_1(A) = V_n(\mathcal{W}_1(A)).$$

DEFINITION 20.2. For any commutative ring A, let us define $I(A)$ to be the submodule of $\mathcal{W}_1(A)$ generated by all the images $V_n(\mathcal{W}_1(A))$, where n is a positive integer which is not divisible by p:

$$I(A) = (\sum_{p \nmid n} V_n(\mathcal{W}_1(A)))$$

Let us denote by $\bar{I}(A)$ its closure. Then we define:

$$\mathcal{W}^{(p)}(A) \stackrel{\text{def}}{=} \mathcal{W}_1(A)/\overline{I(A)}$$

PROPOSITION 20.3. *Let us define a map*

$$\Phi : A^{\mathbb{N}} \ni (a_i) \mapsto \sum_{i=0}^{\infty} (1 - a_i T^{p^i})_W \in \mathcal{W}^{(p)}(A).$$

Then Φ is a bijection.

PROOF. Surjectivity: Every element $(f)_W$ of $\mathcal{W}_1(A)$ may be written as $(f)_W = \sum_j (1 - c_j T^j)_W$. Knowing that $(1 - c_j T^j)_W$ is an element of $I(A)$ whenever j is not divisible by p, we see that $\Phi((c_{p^i})_{i=0}^{\infty}) = (f)_W$. Injectivity: Assume $\Phi((a_i)) = \Phi((b_i))$. Let i_0 be the smallest integer i such that $a_i \neq b_i$. Then by subtraction we obtain an equation

$$\sum_{i \geq i_0} (1 - a_i T^{p^i})_W = \sum_{i \geq i_0} (1 - b_i T^{p^i})_W$$

in $\mathcal{W}^{(p)}(A)$.

$$(1 - a_{i_0} T^{p^{i_0}} + \text{higher order terms})_W = (1 - b_{i_0} T^{p^{i_0}} + \text{higher order terms})_W$$

Since we know that the terms of order p^{i_0} are not affected by additions of elements of $\overline{I(A)}$, we see $a_{i_0} = b_{i_0}$, which is a contradiction.

□