No.8: The ring of p-adic Witt vectors revisited

Temporarary summary of our definitions and results

- ⋆ $\Lambda(A) = 1 + TA[[T]]$.
- ⋆ $(f)_W + (g)_W = (fg)_W$.
- ⋆ $[c] \stackrel{\text{def}}{=} (1 - cT)$.
- ⋆ $[c].(f)_W = (f(cT))_W$.
- ⋆ $V_n((f)_W) = (f(T^n))_W$.
- ⋆ Everyelement x of $\Lambda(A)$ may be uniquely written as

$$x = \sum_{j=1}^{\infty} V_j[x_j] \qquad (\{x_j\} \in A^{\mathbb{Z}_{>0}})$$

  - ∘ $\{x_j\}$ is called the Witt component of x.
- ⋆ $[x] + [y] = \sum_j V_j[\alpha_j(x, y)]$
  - ∘ $\{\alpha_j\}$ is the Witt component of $[x] + [y] =$ $(1-(x+y)T+xy)T^2)_W = (1-uT+vT^2)_W = \sum_{j=1}^{\infty} V_j[\alpha_j(x, y)]$.
  - ∘ $\alpha_1 = u, \alpha_2 = -v, \alpha_3 = -uv, \alpha_4 = -u^2v, \alpha_5 = uv^2-u^3v, \ldots$
- ⋆ $V_n[x] + V_n[y] = \sum_{j=1}^{\infty} V_n[\alpha_j(x, y)]$
- ⋆ $V_n[x] \cdot V_m[y] = d \cdot V_l[x^{m'} y^{n'}]$
  $(d = \gcd(n, m), n = n'd, m = m'd, l = \mathrm{lcm}(n, m))$

For any element x of $\Lambda(A)$, we consider the (set-theoretic) support support(x) of the Witt component $\{x_j\}$ of x.

$$\mathrm{support}(x) = \{j; x_j \neq 0\}$$

- ⋆ $x \in \mathrm{Image}(V_n) \iff \mathrm{support}(x) \in n\mathbb{Z}$
- ⋆ $x \in I_{(p)} \iff \mathrm{support}(x) \in \cup_{q \in S_p} q\mathbb{Z}$ $\quad (S_p = \{q :$ prime $, q \neq p\}$.

We put $P = \{1, p, p^2, p^3, \ldots, \}$. By results of unique factorization, we see:

$$\mathbb{Z}_{>0} = (\cup_{q \in S_p} q\mathbb{Z}_{>0}) \coprod P$$

We put $\Lambda^{(p)}(A) = \Lambda(A)/I_{(p)}$. We have:

Proposition 8.1. *Let p be a prime number. Let A be a ring of characteristic. Then:*

(1) *Every element of $\Lambda^{(p)}(A)$ is written uniquely as*

$$\sum_{j=0}^{\infty} V_p^j([x_j]) \qquad (x_j \in A).$$

(2) *For any $x, y \in A$, we have*

$$V_p^n([x]) \cdot V_p^m([y]) = V_p^{n+m}([x^{p^m} y^{p^n}]).$$

(3) *A map*

$$\varphi : \Lambda^{(p)}(A) \ni \sum_{j=0}^{\infty} V_p^n([x_j]) \mapsto x_0 \in A$$

is a ring homomorphism from $(\Lambda^{(p)}, +, \cdot)$ to $(A, +, \times)$.

(4) $\mathrm{Ker}(\varphi) = \mathrm{Image}(V_p)$.
(5) *An element* $x \in \Lambda^{(p)}$ *is invertible in* $\Lambda^{(p)}$ *if and only if* $\varphi(x)$ *is invertible in* A.

□

COROLLARY 8.2. *If* k *is a field of characteristic* $p \neq 0$*, then* $\Lambda^{(p)}$ *is a local ring with the residue field* k*. If furthermore the field* k *is* **perfect** *(that means, every element of* k *has a* p*-th root in* k*), then every non-zero element of* $\Lambda^{(p)}$ *may be writen as*

$$p^k \cdot x \qquad (k \in \mathbb{N}, x \in (\Lambda^{(p)})^{\cdot} \text{ (i.e. } x\text{:invertible}))$$

Since any integral domain can be embedded into a perfect field, we deduce the following

COROLLARY 8.3. *Let* A *be an integral domain of characteristic* $p \neq 0$*. Then* $\Lambda^{(p)}(A)$ *is an integral domain of characteristic* 0*.*

PROOF. $\Lambda^{(p)}(\iota)$ is always an injection when ι is. □