AFFINE GROUP SCHEMES

YOSHIFUMI TSUCHIMOTO

We basically follow a treatment given in [1]

Definition 0.1. A functor F: (*rings*) $\to$ (*sets*) is said to be representable if there exists a ring A such that

$$F(R) = \mathrm{Hom}(A, R)$$

Examples $\mathrm{GL}_2(R)$, $\mathrm{SL}_2(R)$, $\mathrm{GL}_n(R)$, $\mathrm{SL}_n(R)$ are representable.

Theorem 0.2 (Yoneda's lemma). *Let E and F be set-valued functors erpresented by k-algebras A and B. The natural maps $E \to F$ correspond to k-algebra homomorphisms $B \to A$.*

1. tensor products

tensor products of modules over an algebra

Definition 1.1. Let A be a (not necessarily commutative) ring. Let M be a right A-module. Let N be a left A-module. Then we define the tensor product of M and N over A, denoted by

$$M \otimes_A N$$

as a module generated by symbols

$$\{m \otimes n; m \in M, n \in N\}$$

with the following relations.

(1)
$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n \quad (m_1, m_2 \in M,\ n \in N)$$
(2)
$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2 \quad (m \in M,\ n_1, n_2 \in N)$$
(3)
$$ma \otimes n = m \otimes an \qquad (m \in M,\ n \in N,\ a \in A)$$

universality of tensor products

Definition 1.2. Let A be a (not necessarily commutative) ring. Let M be a right A-module. Let N be a left A-module. Then for any module X, a map $f : M \times N \to X$ is said to be an A-balanced biadditive map if it satisfies the following conditions.

(1) $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n) \quad (\forall m_1, m_2 \in M, \forall n \in N)$
(2) $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2) \quad (\forall m \in M, \forall n_1, n_2 \in N)$
(3) $f(ma, n) = f(m, an) \quad (\forall m \in M, \forall n \in N, \forall a \in A)$

Lemma 1.3. *Let A be a (not necessarily commutative) ring. Let M be a right A-module. Let N be a left A-module. Then for any module X, there is a bijective additive correspondence between the following two objects.*

(1) *An A-balanced bilinear map $M \times N \to X$*
(2) *An additive map $M \otimes_A N \to X$*

1.1. **additional structures on tensor products.**

LEMMA 1.4. *Let A be a (not necessarily commutative) ring. Let M be a right A-module. Let N be a left A-module. If M carries a structure of an A-algebra, then the tensor product $M \times_A N$ carries a structure of M-module in the following manner.*

$$x.(y \otimes n) = (xy) \otimes n \qquad (x, y \in M, n \in N)$$

For $G = \mathrm{GL}_n$ of $G = \mathrm{SL}_n$, the multiplication map $G \times G \to G$, the unit: $\{e\} \to G$, the inverse $G \to G$ are natural maps. The corresponding ring $k[G]$ satisfies a certain set of axioms.

1.2. **bialgebras.**

DEFINITION 1.5. Let K be a field. $(B, m, \eta, \Delta, \epsilon)$ is a bialgebra over K if it has the following properties:

(1) B is a vector space over K;
(2) There are K-linear maps (multiplication) $m : B \otimes B \to B$ (equivalent to K-multilinear map $m : B \times B \to B$) and (unit) $\eta : K \to B$, such that (B, m, η) is a unital associative algebra.
(3) There are K-linear maps (comultiplication) $\Delta : B \to B \to B$ and (counit) $\epsilon : B \to K$, such that (B, Δ, ϵ) is a (counital coassociative) coalgebra.
(4) The pair (m, Δ) satisifies the following compatibility condition. $\Delta(m(f, g)) = (m \otimes m)((1 \otimes \tau \otimes 1)\Delta(f)\Delta(g))$ (where $\tau(b_1 \otimes b_2) = b_2 \otimes b_1$.)

DEFINITION 1.6. A Hopf algebra $(B, m, \eta, \Delta, \epsilon, S)$ is a bialgebra $(B, m, \eta, \Delta, \epsilon)$ with a K-linear map $S : B \to B$ ('antipode') which satisfy the following condition.

$$m(S \otimes 1)\Delta = \eta\epsilon = m(1 \otimes S)\Delta$$

For bialgebras, we denote the product $m(f, g)$ as fg. Furthermore, the coproduct $\Delta(f)$ is a value of the sum of a type $\Delta(f) = \sum_i f^i_{(1)} f^i_{(2)}$, which we simply denote as $f_{(1)} f_{(2)}$ ("sumless version of Sweedler's notation").

EXAMPLE 1.7. $GL_2(K)$. $B = K[x, y, z, w, (xy-zw)^{(-1)}] = K[x, y, z, w | xy - zw \neq 0]$.

$$\Delta(x) = x \otimes x + y \otimes z, \quad \Delta(y) = x \otimes y + z \otimes w,$$
$$\Delta(z) = z \otimes x + w \otimes z, \quad \Delta(w) = z \otimes y + w \otimes w.$$
$$S(x) = w(xw - yz)^{-1} \quad \text{etc.}$$

REFERENCES

[1] W. C. Waterhouse, *Introduction to affine group schemes*, Springer Verlag, 1997.