CONGRUENT ZETA FUNCTIONS. NO.7

YOSHIFUMI TSUCHIMOTO

7.1. **Jacobi symbol.**

DEFINITION 7.1. Let m be a positive odd integer. Let us factor m:

$$m = \prod_i p_i^{e_i}$$

where p_i are primes. Then for any $n \in \mathbb{Z}$, we define Jacobi symbols as follows

$$\left(\frac{n}{m}\right) = \prod_i \left(\frac{n}{p_i}\right)^{e_i}$$

We further define

$$\left(\frac{a}{p}\right) = 0 \text{ if } a \in p\mathbb{Z}.$$

THEOREM 7.2 (quadratic reciprocity theorem). *For any positive odd integers n, m, we have*

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = (-1)^{(m-1)(n-1)/4}.$$

THEOREM 7.3. *Let n be a postive odd integer. Then:*

(1) $\left(\frac{-1}{m}\right) = (-1)^{(m-1)/2}$.
(2) $\left(\frac{2}{m}\right) = (-1)^{(m^2-1)/8}$.

EXERCISE 7.1. $p = 113357$ is a prime. (You may use the fact without proving it.) Is there any integer n such that

$$n^2 = 11351 \text{ in } \mathbb{Z}/p\mathbb{Z}\ ?$$

If so, can you find such n?

A litte appendix. (The following is borrowed from Wikipedia("Riemann zeta function",Japanese version,May 2019))

$$\zeta(s) = \sum_{s=1}^{\infty} \frac{1}{n^s} = \prod (1 - \frac{1}{p^s})^{-1}$$

$$\begin{aligned}
\log(\zeta(s)) &= -\sum_p \log(1 - \frac{1}{p^s}) \\
&= \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{ns}} \\
&= \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{ns}} \\
&= s \sum_{n=1}^{\infty} \frac{1}{n} \sum_p \int_{p^n}^{\infty} x^{-s-1} dx \\
&= s \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} (\sum_p [?p \leq x^{1/n}]) x^{-s-1} dx \\
&= s \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \pi(x^{1/n}) x^{-s-1} dx \\
&= \int_1^{\infty} \Pi(x) x^{-s-1} dx
\end{aligned}$$

here we have put

$$\pi(x) = \#\{p; p \leq x\}, \qquad \Pi(x) = \sum_{n=1}^{\infty} \frac{1}{n} \pi(x^{1/n}).$$

Note also that we have used $\frac{1}{p^{ns}} = s \int_{p^n}^{\infty} x^{-s-1} dx$.