CONGRUENT ZETA FUNCTIONS. NO.3.01

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.01

3.1. **Introduction.** For any topological space X, we define

$$C(X) = \{X \to \mathbb{C};\ \text{continuous}\}.$$

It has a natural structure of a ring by introducing “point-wise operations”:

$$(f+g)(x) = f(x)+g(x), \quad (f\cdot g)(x) = f(x)\cdot g(x) \quad (\forall x \in X, \forall f, g \in C(X)).$$

It has an extra structure of $*$-operation:

$$(f^*)(x) = \overline{f(x)} \qquad (complexconjugate).$$

and a topology (locally uniform topology) which we shall not describe in detail.

Theorem 3.1 (Gelfand-Naimark). *(”Commutative Case”)*

$$\text{(Compact Hausdorff space)} \ni K \mapsto C(K) \in (C^*\text{-algebras})$$

is a bijection.

The inverse of the correspondence above is given by associating to a commutative C^*-algebra A a set

$$\operatorname{Spm}(A) = \{\text{maximal ideal of } A\}$$

with a certain topology.

A first interesting part of modern algebraic geometry is that we may mimic the correspondence in the Gelfand-Naimark theorem above and associate to any commutive ring a compact (but not Hausdorff) space $\operatorname{Spec}(A)$. The elements of A may then be considered as “continuous functions” on $\operatorname{Spec}(A)$.

The upshot is that we may “cut and paste”, as one usually does with functions, elements of abstract commutative rings. Any other method of functional analysis also has the possibility to be applied in the commutative ring theory.

On the other hand, it is possible to manipulate the compact space $\operatorname{Spec}(A)$ and create new algebras out of the existing commutative ring A. We may furtheremore paste such $\operatorname{Spec}(A)$'s altogther and define another geometric objects.

Problem 3.2. Let X be a finite set with the discrete topology. Show that $C(X)$ has exactly $\#X$ pieces of maximal ideals.

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3.2. Spec A.

Definition 3.3. An ideal I of a ring A is said to be

(1) a prime ideal if A/I is an integral domain.
(2) a maximal ideal if A/I is a field.

Definition 3.4. Let A be a ring. Then we define its *affine spectrum* as

$$\operatorname{Spec}(A) = \{\mathfrak{p} \subset A; \mathfrak{p} \text{ is a prime ideal of } A\}.$$

Definition 3.5. Let A be a ring. For any $\mathfrak{p} \in \operatorname{Spec}(A)$ we define "evaluation map" $\mathrm{eval}_\mathfrak{p}$ as follows:

$$\mathrm{eval}_\mathfrak{p} : A \to A/\mathfrak{p}$$

Note that $A/\mathfrak{p}$ is a subring of a field $Q(A/\mathfrak{p})$, the field of fractions of the integral domain $A/\mathfrak{p}$. We interpret each element f of A as a something of a "fuction", whose value at a point $\mathfrak{p}$ is given by $\mathrm{eval}_\mathfrak{p}(f)$.

We introduce a topology on $\operatorname{Spec}(A)$. We basically mimic the following Lemma:

Lemma 3.6. *Let X be a topological space. then for any continuous function $f : X \to \mathbb{C}$, its zero points $\{x \in X; f(x) = 0\}$ is a closed subset of X. Furthermore, for any family $\{f_\lambda\}$ of continous $\mathbb{C}$-valued functions, its common zeros $\{x \in X; f_\lambda(x) = 0 \quad (\forall\lambda)\}$ is a closed subset of X.*

Definition 3.7. Let A be a ring. Let S be a subset of A, then we define the common zero of S as

$$V(S) = \{\mathfrak{p} \in \operatorname{Spec}(A); \mathrm{eval}_\mathfrak{p}(f) = 0 \qquad (\forall f \in S\}.$$

For any subset S of A, let us denote by $\langle S\rangle_A$ the ideal of A generated by S. Then we may soon see that we have $V(S) = V(\langle S\rangle_A)$. So when thinking of $V(S)$ we may in most cases assume that S is an ideal of A.

Lemma 3.8. *Let A be a ring. Then:*

(1) $V(0) = \operatorname{Spec}(A), \quad V(\{1\})(= V(A)) = \emptyset$.
(2) *For any family $\{I_\lambda\}$ of ideals of A, we have $\cap_\lambda V(I_\lambda) = V(\sum_\lambda I_\lambda)$.*
(3) *For any ideals I, J of A, we have $V(I) \cup V(J) = V(I \cdot J)$.*

Proposition 3.9. *Let A be a ring. $\{V(I); I$ is an ideal of $A\}$ satisfies the axiom of closed sets of* $\operatorname{Spec}(A)$. *We call this the Zariski topology of* $\operatorname{Spec}(A)$.

Problem 3.10. Prove Lemma 3.8.

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3.3. Examples of $\operatorname{Spec} A$.

(1) $\operatorname{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) | p: \text{ prime }\}$.
(2) $\operatorname{Spec}(\Bbbk[X]) = \{(0)\} \cup \{(p) | p: \text{ irreducible polynomial}\}$.
(3) $\operatorname{Spec}(\mathbb{C}[X, Y]) = \{(0)\} \cup \{(p) | p: \text{ irreducible polynomial}\} \cup \{(X - a, Y - b) | a, b \in \mathbb{C}\}$.

3.4. Further properties of Spec.

Lemma 3.11. *Let A be a ring. Then:*

(1) *For any $f \in A$, $D(f) = \{\mathfrak{p} \in \operatorname{Spec}(A); f \notin \mathfrak{p}\}$ is an open set of* $\operatorname{Spec}(A)$.
(2) *Given a point $\mathfrak{p}$ of* $\operatorname{Spec}(A)$ *and an open set U which contains $\mathfrak{p}$, we may always find an element $f \in A$ such that $\mathfrak{p} \in D(f) \subset U$. (In other words, $\{D(f)\}$ forms an open base of the Zariski topology.*

Theorem 3.12. *For any ring A,* $\operatorname{Spec}(A)$ *is compact. (But it is not Hausdorff in most of the case.)*

DEFINITION 3.13. Let X be a topological space. A closed set F of X is said to be **reducible** if there exist closed sets F_1 and F_2 such that

$$F = F_1 \cup F_2, \quad F_1 \neq F, F_2 \neq F$$

holds. F is said to be **irreducible** if it is not reducible.

Recall that we have defined, for any ring A and for any ideal I, a closed subset $V(I)$ of $\operatorname{Spec}(A)$ by

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(A); \operatorname{eval}_{\mathfrak{p}}(f) = 0 \quad (\forall f \in I).$$

We define:

DEFINITION 3.14. Let A be a ring. Let X be a subset of $\operatorname{Spec}(A)$. Then we define

$$I(X) = \{f \in A; \operatorname{eval}_{\mathfrak{p}}(f) = 0 \quad (\forall \mathfrak{p} \in I)\}.$$

LEMMA 3.15. *Let A be a ring. Then:*

1. *For any subset X of* $\operatorname{Spec}(A)$, *$I(X)$ is an ideal of A.*
2. *(For any subset S of A, $V(S)$ is a closed subset of* $\operatorname{Spec}(A)$.*)*
3. *For any subsets $X_1 \subset X_2$ of* $\operatorname{Spec}(A)$, *we have $I(X_1) \supset I(X_2)$.*
4. *For any subsets $S_1 \subset S_2$ of A, we have $V(S_1) \supset V(S_2)$.*
5. *For any subset X of* $\operatorname{Spec}(A)$, *we have $V(I(X)) \subset X$.*
6. *For any subset S of A, we have $I(V(S)) \subset S$.*

COROLLARY 3.16. *Let A be a ring. Then:*

1. *For any subset X of* $\operatorname{Spec}(A)$, *we have $I(V(I(X))) = I(X)$.*
2. *For any subset S of A, we have $V(I(V(S))) = V(S)$.*

DEFINITION 3.17. Let I be an ideal of a ring A. Then we define its **radical** to be

$$\sqrt{I} = \{x \in A; \exists N \in \mathbb{Z}_{>0} \text{ such that } x^N \in I\}.$$

PROPOSITION 3.18. *Let A be a ring. Then;*

1. *For any ideal I of A, we have $V(I) = V(\sqrt{I})$.*
2. *For two ideals I, J of A, $V(I) = V(J)$ holds if and only if $\sqrt{I} = \sqrt{J}$.*
3. *For an ideal I of A, $V(I)$ is irreducible if and only if $\sqrt{I}$ is a prime ideal.*