抽象代数学特論

8:50 ~

10進法 → $1234.0 \leftrightarrow [0.4321]_{10}$

$1230 \leftrightarrow [0.0321]_{10}$

$1200 \leftrightarrow [0.0021]_{10}$

$2200 \leftrightarrow [0.0022]_{10}$

「有限小数」

0

$\mathbb{Z}_p$:「無限小数」を考える

$\cdots 432200 \leftrightarrow [0.002234567\,8\cdots]_{10}$

整数の対応

$\mathbb{Q}$ の通常の世界

1.4, 1.41, 1.414, 1.4141, ⋯ : $\mathbb{Q}$ の Cauchy列

極限 $\sqrt{2} \notin \mathbb{Q}$

$\mathbb{Q}$ は完備ではない。
complete

↓ 完備化

$\mathbb{R}$

$(X, d) \xhookrightarrow[\text{dense.}]{} (\bar{X}, d)$

complete metric space

$\bar{X}$の作り方. ① XのCauchy列の全体を考える

$$\mathrm{Cauchy}(X) = \{ \underset{\text{Cauchy}}{\{a_i\}_{i=1}^{\infty} ; a_i \in X} \} \subset X^{(\mathbb{Z}_{>0})}$$

Cauchy(X)のギキョリ(pseudo-distance)を入れる.

$$d_1(\underset{\{x_i\}_{i=1}^{\infty}}{x}, \underset{\{y_i\}_{i=1}^{\infty}}{y}) = \lim_{i \to \infty} d(x_i, y_i)$$

: 極限が存在, ギキョリとなる.

($d_1(x,y) = 0 \Rightarrow x = y$ とは限らない「キョリ」)

Cauchy(X)に同値関係∼を

$x \sim y \Leftrightarrow d_1(x, y) = 0$ で入れる.

⓪ ∼は同値関係

① $\boxed{\mathrm{Cauchy}(X)/\sim}$ にょり

$d_2((x\text{のクラス}), (y\text{のクラス})) \underset{\text{def}}{=} d_1(x, y)$

でwell-definedなキョリが入る

② $\mathrm{Cauchy}(X)/\sim$ は完備 (w.r.t. d_2)

Ex. 2.13

$$(X, d) \hookrightarrow (\overline{X}, d)$$

完備化の1つ.

$$\begin{tikzcd}
(X, d) \arrow[r, hook] & (\overline{X}, d) \arrow[d, dash] \\
 & \left(\mathrm{Cauchy}(X)/_{\sim}, d_2\right)
\end{tikzcd}$$

$$\begin{tikzcd}
 & \iota_2(x_i) \arrow[r, squiggly] \arrow[d, phantom, "\cap"] & \bigcirc \\
 & (\overline{X}_2, d_2) \arrow[d, shift right] \arrow[d, phantom, "\cong"] & \\
(X, d) \arrow[r, hook, "\text{dense}"'] \arrow[ur, bend left, "\text{dense}", "\iota_2"'] & (\overline{X}_1, d_1) \arrow[u, shift right] \arrow[d, phantom, "\cup"] & \\
\{x_i\} \arrow[r, "\text{収束}"'] & x \arrow[uuur] &
\end{tikzcd}$$

Ex 2.C 完備化空間の証明の詳細をのべよ.

完備化定理

$$\begin{tikzcd}
 & (Y, \tilde{d}) \text{ complete} \\
(X, d) \arrow[r, hook, "\text{dense}"'] \arrow[ur, hook, "\text{dense}"] & (\overline{X}, \overline{d}) \text{ complete} \arrow[u, "\wr\ \text{等長}\ \text{unique}"']
\end{tikzcd}$$

(X, d) の完備化
completion

$$(\mathbb{Z}, |\cdot|_p) \hookrightarrow (\overline{\mathbb{Z}}, |\cdot|_p)$$

$\mathbb{Z}_p$ と書く. p進整数環とよぶ. Thm 2.7

Def 2.6