CONGRUENT ZETA FUNCTIONS. NO.13

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Let us site wikipedia:
(`https://en.wikipedia.org/wiki/Lefschetz_fixed-point_theorem` with some modifications by Tsuchimoto–consult the original page for details)

THEOREM 13.1. *(Lefschetz) Let*

$$f : X \to X$$

be a continuous map from a compact triangulable space X *to itself. Define the Lefschetz number* Λ_f *of* f *by*

$$\Lambda_f := \sum_{k \geq 0} (-1)^k \operatorname{tr}(f_* | H_k(X, \mathbb{Q})),$$

the alternating (finite) sum of the matrix traces of the linear maps induced by f *on* $H_k(X, \mathbb{Q})$*, the singular homology groups of* X *with rational coefficients. A simple version of the Lefschetz fixed-point theorem states: if* $\Lambda_f \neq 0$*, then* f *has at least one fixed point, i.e., there exists at least one* x *in* X *such that* $f(x) = x$*. In fact, since the Lefschetz number has been defined at the homology level, the conclusion can be extended to say that any map homotopic to* f *has a fixed point as well.*

A stronger form of the theorem, also known as the Lefschetz-Hopf theorem, states that, if f *has only finitely many fixed points, then*

$$\sum_{x \in \mathrm{Fix}(f)} i(f, x) = \Lambda_f,$$

where $\mathrm{Fix}(f)$ *is the set of fixed points of* f*, and* $i(f, x)$ *denotes the index of the fixed point* x*. From this theorem one deduces the Poincaré-Hopf theorem for vector fields*