CONGRUENT ZETA FUNCTIONS. NO.4

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4.1. **Definition of congruent Zeta function.**

DEFINITION 4.1. Let q be a power of a prime. Let $\{f_1, f_2, \dots, f_m\}$ be a set of polynomial equations in n-variables over $\mathbb{F}_q$. Recall that we have defined in section 2 the affine variety $V = \operatorname{Spec}(\mathbb{F}_q[X_1, \dots, X_n]/(f_1, \dots, f_m)$. We may identify $V(\mathbb{F}_{q^s})$ with the set of solutions of $\{f_1, \dots, f_m\}$ in $(\mathbb{F}_{q^s})^n$. That means,

$$V(\mathbb{F}_{q^s}) = \{x \in (\mathbb{F}_{q^s})^n; f_1(x) = 0, f_2(x) = 0, \dots, f_m(x) = 0\}.$$

Then we define

$$Z(V/\mathbb{F}_q, T) = \exp(\sum_{s=1}^{\infty}(\frac{1}{s}\#V(\mathbb{F}_{q^s})T^s)).$$

EXERCISE 4.1. Compute congruent zeta function for $V = \operatorname{Spec}(\mathbb{F}_q[X, Y](XY))$.

EXERCISE 4.2. Compute congruent zeta function for $V = \operatorname{Spec}(\mathbb{F}_q[X, Y]/(X^2 + Y^2 - 1))$.

4.2. **First properties of congruent Zeta function.**

Let us first recall an elementary formula

LEMMA 4.2.

$$\sum_{k=1}^{\infty} \frac{1}{k} T^k = -\log(1 - T)$$

DEFINITION 4.3. Let $\Bbbk$ be a ring. We define $\mathbb{A}^n$ as the affine spectrum of the polynomial ring $\Bbbk[X_1, \dots, X_n]$. For any field (or ring) L over $\Bbbk$, we have

$$\mathbb{A}^n(L) = \{(x_1, x_2, \dots, x_n); x_1, x_2, \dots, x_n \in L\}.$$

PROPOSITION 4.4.

$$Z(\mathbb{A}^n/\mathbb{F}_q, T) = \frac{1}{1 - q^n T}$$

PROPOSITION 4.5. *Let V, W, W_1, W_2 be affine varieties.*

1. *If $\#V(\mathbb{F}_{q^s}) = \#W(\mathbb{F}_{q^s})$ for any s ,then $Z(V/\mathbb{F}_q, T) = Z(W/\mathbb{F}_q, T)$.*
2. *If $\#V(\mathbb{F}_{q^s}) = \#W_1(\mathbb{F}_{q^s}) + \#W_2(\mathbb{F}_{q^s})$ for any s ,then:*

$$Z(V/\mathbb{F}_q, T) = Z(W_1/\mathbb{F}_q, T) Z(W_2/\mathbb{F}_q, T).$$

PROPOSITION 4.6. *Let $f \in \mathbb{F}_q[X]$ be an irreducible polynomial in one variable of degree d. Let us consider $V = \operatorname{Spec}(\Bbbk[X]/(f)$. Then:*

(1)

$$V(\mathbb{F}_{q^s}) = \begin{cases} d & \text{if } d|s \\ 0 & \text{otherwise} \end{cases}$$

(2)

$$Z(V/\mathbb{F}_q, T) = \frac{1}{1 - T^d}$$

EXERCISE 4.3. Describe what happens when we omit the assumption of f being irreducible in Proposition 4.6.