ALGEBRAIC GEOMETRY AND RING THEORY

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curves (over $\mathbb{C}$)

Let C be a curve over $\mathbb{C}$. A divisor D is a formal finite sum $\sum n_i P_i$ of points P_i on the curve C. For any such divisor, we may consider a sheaf $\mathcal{O}(D)$. We call the sum $\sum_i n_i$ the *degree* of D. It is also refered to as the degree of $\mathcal{O}(D)$.

An $\mathcal{O}$-module $\mathcal{F}$ on C is called *invertible* if it is locally isomorphic to the structure sheaf $\mathcal{O}$. Any invertible sheaf is actually isomorphic to a sheaf $\mathcal{O}(D)$ for some divisor D.

A divisor $D = \sum n_i P_i$ is called effective if $n_i \geq 0$ for all i. For any invertible sheaf $\mathcal{F}$ over C, we have a exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}(D) \to \mathcal{F}(D)/\mathcal{F} \to 0. \quad \text{: exact}$$

We have thus the associated long exact sequence on cohomology:

$$\begin{aligned} 0 &\to H^0(\mathcal{F}) \to H^0(\mathcal{F}(D)) \to H^0(\mathcal{F}(D)/\mathcal{F}) \\ &\to H^1(\mathcal{F}) \to H^1(\mathcal{F}(D)) \to 0. \end{aligned}$$

We should also mention the genus $g(C)$ of the curve. It is topologically the "number of holes" of the surface $C(\mathbb{C})$.

Theorem 10.1 (Riemann-Roch). *Let C be a non-singular projective curve over $\mathbb{C}$. For any invertible sheaf $\mathcal{F}$ on C, we have*

$$\dim H^0(\mathcal{F}) - \dim H^1(\mathcal{F}) = 1 - g + \deg(\mathcal{F})$$

We have an important sheaf $\omega = \Omega^1$ on C. For any $\mathcal{O}$-module $\mathcal{F}$ on C, we may consider the sheaf of $\mathcal{F}$-valued 1-forms $\mathcal{F} \otimes \omega$.

We also note that for any invertible sheaf $\mathcal{F}$ on C, we have its dual $\mathcal{F}^\vee$:

$$\mathcal{O}(D)^\vee = \mathcal{O}(-D).$$

Theorem 10.2 (Serre duality).

$$H^i(\mathcal{F})^\vee \cong H^{1-i}(\mathcal{F}^\vee \otimes \omega)$$

We may understand the situation of the two theorems above by using a "formal version of the Čech cohomology". Namely, for any point P of C with a local coordinate t such that $t(P) = 0$, We define formal-Spec($\mathbb{C}[[t]]$) as a formal "neighbourhood" of P. C may then be covered as

$$C = C \setminus \{P_1, \ldots, P_n\} \cup U = \dot{C} \cup U$$

where U is the union of such formal "neighbourhoods" of P_i's. One may then mimic the Čech cohomology and obtain a Čech complex. Namely, for any $\mathcal{O}$-module $\mathcal{F}$ on C, we have a complex

$$\mathcal{F}(\dot{C}) \oplus \mathcal{F}(U) \to \mathcal{F}(\dot{U})$$

whose cohomologies are isomorphic to $H^\bullet(X; \mathcal{F})$. If $\mathcal{F}$ is invertible, we have also a residue pairing

$$F(\dot{U}) \times \mathcal{F}^\vee(\dot{U}) \to \mathbb{C}$$

which gives rise to the Serre duality.