ALGEBRAIC GEOMETRY AND RING THEORY

YOSHIFUMI TSUCHIMOTO

quadratic and cubic curves

Definition 8.1. For any ring A, we define its Krull dimension to be the maximum of ascending chains of primes in A. Namely,

$$\mathrm{Krulldim}(A) = \max\{n; A \supset \mathfrak{p}_n \supsetneq \mathfrak{p}_{n-1} \supsetneq \cdots \supsetneq \mathfrak{p}_0\}$$

Definition 8.2. A local ring $A = (A, \mathfrak{m})$ is called regular if its Krull dimension is equal to $\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$¿

Proposition 8.3. *Let* $\Bbbk$ *be a field of characterictic* $\neq 2$. *Then every quadratic curve (a curve defined by a homogeneous polynomial of degree* 2*) in* $\mathbb{P}^2$ *over* $\Bbbk$ *is isomorphic to a curve of the form*

$$aX^2 + bY^2 + cZ^2 = 0. \qquad (a, b, c \in \Bbbk).$$

In particular, every quadratic curve in $\mathbb{P}^2$ *over* $\mathbb{R}$ *is isomorphic to a curve* $X^2 + Y^2 = Z^2$.

Proposition 8.4. *Let* $\Bbbk$ *be a field of characterictic* $\neq 2, 3$. *Then every cubic curve (a curve defined by a homogeneous polynomial of degree* 3*) in* $\mathbb{P}^2$ *over* $\Bbbk$ *is isomorphic to a curve of the form*

$$ZY^2 = X^3 + pXZ^2 + qZ^3 \qquad (p, q \in \Bbbk).$$

It should be meaningful to point out:

Proposition 8.5. *Let* τ *be a imaginary number in* $\mathbb{C}$. τ *defines a lattice (a discrete subgroup of rank* 2 *in* $\mathbb{C}$*)* $L = \mathbb{Z} + \tau\mathbb{Z}$. *A complex manifold* $\mathbb{C}/L$ *may be embedded to the complex projective plane* $\mathbb{P}^2(\mathbb{C})$ *by the Weierstrass* $\wp$*-function* $\wp(z; L)$ *and its derivative* $\wp'(z; L)$. *Namely, a rational map defined by*

$$\mathbb{C} \ni z \mapsto [\wp(z; L) : \wp'(z; L) : 1] \in \mathbb{P}^2(\mathbb{C})$$

gives a holomorphic map $f : \mathbb{C}/L \to \mathbb{P}^2(\mathbb{C})$. *moreover,* $\wp, \wp'$ *satisfy a cubic relation so that* f *gives an isomorphism of* $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ *and a cubic curve in* $\mathbb{P}^2(\mathbb{C})$.