ALGEBRAIC GEOMETRY AND RING THEORY

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5.1. **general localization of a commutative ring.** We define a localization of a commutative ring in a more general situation than in subsection **??**.

Definition 5.1. Let A be a commutative ring. Let S be its subset. We say that S is multiplicative if

(1) $1 \in S$
(2) $x, y \in S \implies xy \in S$

holds.

Definition 5.2. Let S be a multiplicative subset of a commutative ring A. Then we define $A[S^{-1}]$ as

$$A[\{X_s; s \in S\}]/(\{sX_s - 1; s \in S\})$$

where in the above notation X_s is a indeterminate prepared for each element $s \in S$.) We denote by ι_S a canonical map $A \to A[S^{-1}]$.

Lemma 5.3. *Let S be a multiplicative subset of a commutative ring A. Then the ring $B = A[S^{-1}]$ is characterized by the following property:*
Let C be a ring, $\varphi : A \to C$ be a ring homomorphism such that $\varphi(s)$ is invertible in C for any $s \in S$. Then there exists a unique ring homomorphism $\psi = \phi[S^{-1}] : B \to C$ such that

$$\varphi = \psi \circ \iota_S$$

holds.

Corollary 5.4. *Let S be a multiplicative subset of a commutative ring A. Let I be an ideal of A given by*

$$I = \{x \in I; \exists s \in S \text{ such that } sx = 0\}$$

Then (1) I is an ideal of A. Let us put $\bar{A} = A/I$, $\pi : A \to \bar{A}$ the canonical projection. Then:
(2) $\bar{S} = \pi(S)$ is multiplicatively closed.
(3) We have

$$A[S^{-1}] \cong \bar{A}[\bar{S}^{-1}]$$

(4)$\iota_{\bar{S}} : \bar{A} \to \bar{A}[\bar{S}^{-1}]$ is injective.

Example 5.5. $A_f = A[S^{-1}]$ for $S = \{1, f, f^2, f^3, f^4, \dots\}$. The total ring of quotients $Q(A)$ is defined as $A[S^{-1}]$ for

$$S = \{x \in A; x \text{ is not a zero divisor of A}\}.$$

When A is an integral domain, then $Q(A)$ is the field of quotients of A.

Definition 5.6. Let A be a commutative ring. Let $\mathfrak{p}$ be its prime ideal. Then we define the localization of A with respect to $\mathfrak{p}$ by

$$A_{\mathfrak{p}} = A[(A \setminus \mathfrak{p})^{-1}]$$

5.2. **general localization of modules.**

DEFINITION 5.7. Let S be a multiplicative subset of a commutative ring A. Let M be an A-module we may define $S^{-1}M$ as

$$\{(m/s); m \in M, s \in S\}/\sim$$

where the equivalence relation $\sim$ is defined by

$$(m_1/s_1) \sim (m_2/s_2) \iff t(m_1s_2 - m_2s_1) = 0 \quad (\exists t \in S).$$

We may introduce a $S^{-1}A$-module structure on $S^{-1}M$ in an obvious manner.

$S^{-1}M$ thus constructed satisfies an universality condition which the reader may easily guess.

5.3. **local rings.**

DEFINITION 5.8. A commutative ring A is said to be a local ring if it has only one maximal ideal.

EXAMPLE 5.9. We give examples of local rings here.

- Any field is a local ring.
- For any commutative ring A and for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$, the localization $A_\mathfrak{p}$ is a local ring with the maximal ideal $\mathfrak{p}A_\mathfrak{p}$.

LEMMA 5.10. (1) *Let A be a local ring. Then the maximal ideal of A coincides with $A \setminus A^\times$.*
(2) *A commutative ring A is a local ring if and only if the set $A \setminus A^\times$ of non-units of A forms an ideal of A.*

PROOF. (1) Assume A is a local ring with the maximal ideal $\mathfrak{m}$. Then for any element $f \in A \setminus A^\times$, an ideal $I = fA + \mathfrak{m}$ is an ideal of A. By Zorn's lemma, we know that I is contained in a maximal ideal of A. From the assumption, the maximal ideal should be $\mathfrak{m}$. Therefore, we have

$$fA \subset \mathfrak{m}$$

which shows that

$$A \setminus A^\times \subset \mathfrak{m}.$$

The converse inclusion being obvious (why?), we have

$$A \setminus A^\times = \mathfrak{m}.$$

(2) The "only if" part is an easy corollary of (1). The "if" part is also easy.

□

COROLLARY 5.11. *Let A be a commutative ring. Let $\mathfrak{p}$ its prime ideal. Then $A_\mathfrak{p}$ is a local ring with the only maximal ideal $\mathfrak{p}A_\mathfrak{p}$.*

PROPOSITION 5.12. *Let A be a commutative ring. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ then the stalk $\mathcal{O}_\mathfrak{p}$ of $\mathcal{O}$ on $\mathfrak{p}$ is isomorphic to $A_\mathfrak{p}$.*

DEFINITION 5.13. Let A, B be local rings with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$ respectively. A local homomorphism $\varphi : A \to B$ is a homomorphism which preserves maximal ideals. That means, a homomorphism φ is said to be local if

$$\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$$