ZETA FUNCTIONS. NO.8

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DEFINITION 8.1. A **category** $\mathcal{C}$ is a collection of the following data

(1) A collection $\mathrm{Ob}(\mathcal{C})$ of **objects** of $\mathcal{C}$.
(2) For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set
$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$
of **morphisms**.
(3) For each triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map("composition (rule)")
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

satisfying the following axioms

(1) $\mathrm{Hom}(X, Y) \cap \mathrm{Hom}(Z, W) = \emptyset$ unless $(X, Y) = (Z, W)$.
(2) (Existence of an identity) For any $X \in \mathrm{Ob}(\mathcal{C})$, there exists an element $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ such that
$$\mathrm{id}_X \circ f = f, \quad g \circ \mathrm{id}_X = g$$
holds for any $f \in \mathrm{Hom}(S, X), g \in \mathrm{Hom}(X, T)$ $(\forall S, T \in \mathrm{Ob}(\mathcal{C}))$.
(3) (Associativity) For any objects $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$, and for any morphisms $f \in \mathrm{Hom}(X, Y), g \in \mathrm{Hom}(Y, Z), h \in \mathrm{Hom}(Z, W)$, we have
$$(f \circ g) \circ h = f \circ (g \circ h).$$

DEFINITION 8.2. (1) A morphism $f : X \to Y$ in a category is said to be **monic** if for any object Z of $\mathcal{C}$ and for any morphism $g_1, g_2 : Z \to X$, we have
$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$
(2) A morphism $f : X \to Y$ in a category is said to be **epic** if for any object Z of $\mathcal{C}$ and for any morphism $g_1, g_2 : Y \to Z$, we have
$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2$$

DEFINITION 8.3. An object X is called **initial** (resp. **terminal**) if $\mathrm{Hom}(X, Y)$(*resp.* $\mathrm{Hom}(Y, X)$) consists of only one element for every object Y. We say that an object X is a zero object if X is initial and terminal. It follows that all the zero objects of $\mathcal{C}$ are isomorphic.

DEFINITION 8.4. Let $\mathcal{C}$ be a category with a zero object. We say that an object $X \in \mathrm{Ob}(\mathcal{C})$ is **simple** when $\mathrm{Hom}(X, Y)$ is consisting of monomorph isms and zero-morphisms for every object Y. The **norm** $N(X)$ of an object X is defined as

$$N(X) = \#\,\mathrm{End}(X) = \#Hom(X, X)$$

where $\#\,\mathrm{End}(X)$ is the cardinality of endomorphisms of X. We say that a non-zero object X is **finite** if $N(X)$ is finite.

The treatment here is based on a paper of Kurokawa[1].

DEFINITION 8.5. We denote by $P(C)$ the isomorphism classes of all finite simple objects of C. Remark that for each $P = [X] \in P(C)$ the norm $N(P) = N(X)$ is well-defined, We define the zeta function $(s, \mathcal{C})$ of $\mathcal{C}$ as

$$\zeta(s, C) = \prod_{p \in P(\mathcal{C})} (1 - N(p)^{-s})^{-1}.$$

PROPOSITION 8.6. *The zeta function of the category Ab of abelian groups is equal to the Riemann zeta function. In other words, we have*

$$\zeta(s, \mathrm{Ab}) = \zeta(s).$$

REFERENCES

[1] N. Kurokawa, *Zeta functions of categories*, proc.japan.acad **72** (1996), no. 10, 221–222.