# ZETA FUNCTIONS

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## 1.1. **Introduction.**

$$S^1 = \mathbb{R}/2\pi\mathbb{Z}$$

Fourier analysis tells us that the function space on $S^1$ is topologically spanned by

$$\{f_k(x) = e^{\sqrt{-1}kx}; k \in \mathbb{Z}\}.$$

Let us consider the Laplacian $\Delta = -(\partial/\partial x)^2$.

$$\Delta(f_k(x)) = k^2 f_k(x).$$

$$\Delta = \operatorname{diag}(\dots, 3^2, 2^2, 1^2, 0^2, 1^2, 2^2, 3^2, \dots)$$

The 0-eigenspace of the Laplacian corresponds to the space of constants $H^0(S^1, \mathbb{R})$. We omit it and consider somewhat reduced matrix:

$$\tilde{\Delta} = \operatorname{diag}(\dots, 3^2, 2^2, 1^2, 1^2, 2^2, 3^2, \dots).$$

We may then consider its $-s/2$-th power:

$$\tilde{\Delta}^{-s/2} = \operatorname{diag}(\dots, 3^{-s}, 2^{-s}, 1^{-s}, 1^{-s}, 2^{-s}, 3^{-s}, \dots).$$

Its trace is equal to the Riemann Zeta function (up to the multiplicative constant 2.)

$$\operatorname{tr} \tilde{\Delta}^{-s/2} = 2\sum_{n=1}^{\infty} \frac{1}{n^s} = 2\zeta(s).$$

Remark 1.1. For any compact Riemannian manifold $M$, we may mimic the argument. Namely,

(1) There are Laplacians $\Delta_i$ on the space $\Omega^i(M, \mathbb{C})$ of $i$-forms for $i = 0, 1, 2, \dots, \dim M$.
(2) For each $i$, the 0-eigenspace of $\Delta_i$ is equal to the $i$-th cohomology
(3) For each $i$, we may consider reduced Laplacian $\tilde{\Delta}_i$. It is an self adjoint operator with positive eigenvalues.
(4) The trace of the power $\Delta_i^{-s/2}$ may be considered as the "$i$-th Zeta function of $M$".

Remark 1.2. Eigen vectors of the Laplacians are "particles" on the manifold $M$. The zeta functions are then considered to be "generating functions of the number of particles on $M$".

In general, we employ the following principle:

The zeta functions are "generating functions of number of particles"

The meaning of the term "particles" may vary.

### 1.2. **Formal power series.**

DEFINITION 1.3. Let $A$ be a commutative ring. Let $X$ be a variable. A formal power series in $X$ over $A$ is a formal sum

$$\sum_{i=0}^{\infty} a_i X^i \quad (a_i \in A)$$

We denote by $A[[X]]$ the ring of formal power series in $X$. Namely,

$$A[[X]] = \{\sum_{i=0}^{\infty} a_i X^i; a_i \in A\}.$$

For any element $f = \sum_n a_n X^n$ of $A[[X]]$, we define its **order** as follows:

$$\mathrm{ord}(f) = \inf\{n; a_n \neq 0\}.$$

Then we may define a metric on $A[[X]]$.

$$d(f,g) = \frac{1}{2^{\mathrm{ord}(f-g)}}$$

EXERCISE 1.1. Show that $(A[[X]], d)$ is a complete metric space.

EXERCISE 1.2. Show that $A[[X]]$ is a topological ring. That means, it is a topological space equipped with a ring structure and operations (the addition and the multiplication) is continuous.

### 1.3. **generating functions.**

A generating function is a formal power series in one indeterminate, whose coefficients encode information about a sequence of numbers $\{a_n\}$ that is indexed by the natural numbers.

#### 1.3.1. *Ordinary generating function.*

$$G_0(\{a_n\}; X) = \sum_{n=0}^{\infty} a_n X^n$$

EXAMPLE 1.4. (Examples of ordinary generating functions)

(1) A generating function of a geometric progression:

$$\sum_{i=0}^{n} a^n X^n = \frac{1}{1-aX}.$$

(2) A generating function of an arithmetic progression:

$$\sum_{i=0}^{n} (n+1) X^n = \frac{1}{(1-X)^2}.$$

#### 1.3.2. *Dirichlet series generating function.*

$$G_1(\{a_n\}; s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

PROPOSITION 1.5. *(Euler product expression) Assume* $\{a_n\}$ *is* **multiplicative** *in the sense that*

$$\gcd(n,m) = 1 \implies a_n a_m = a_{nm}$$

*holds, Then we have*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p;prime} \left( \sum_{k=0}^{\infty} \frac{a_{p^k}}{p^{ks}} \right).$$