RESOLUTIONS OF SINGULARITIES.

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05. supplement

Proposition 5.1. *For any ring A, the map $\mathbb{P}^n(A) \to \operatorname{Spec} A$ is proper.*

Proof. See http://amathew.wordpress.com/2010/10/23/a-projective-morphism-is-proper/. □

Corollary 5.2. *For any ring A and for any $\mathbb{N}$-graded ring $S = \oplus_n S_n$ which is generated by a finite subset of S_1 over the ring $S_0 \cong A$, the map $\operatorname{Proj}(S) \to \operatorname{Spec} A$ is proper.*

Proposition 5.3. *Let I be an ideal of a ring A generated by elements $f_0, \dots f_s$. Assume that each of the elements $f_0, f_1, \dots, f_s$ is not a zero divisor in A. Then:*

(1) *$\oplus_{n\in\mathbb{N}} I^n \subset \oplus_{n\in\mathbb{N}} A$ is isomorphic to $A[I \cdot T] \subset A[T]$ for an indeterminate T.*

(2)
$$\operatorname{Proj}(\oplus A[I\dot{T}]) = \operatorname{Proj}(\oplus A[f_0T, f_1T, \dots, f_sT]) = \cup_{i=0}^{s} D_+(f_iT)$$

(3) $D_+(f_iT) \cong \operatorname{Spec} A[f_0/f_i, f_1/f_i, \dots, f_s/f_i]$.

Definition 5.4. Let A be a commutative ring. Let $\mathfrak{p}$ be its prime ideal. Then we define the localization of A with respect to $\mathfrak{p}$ by

$$A_\mathfrak{p} = A[(A \setminus \mathfrak{p})^{-1}]$$

Definition 5.5. A commutative ring A is said to be a local ring if it has only one maximal ideal.

Example 5.6. We give examples of local rings here.

- Any field is a local ring.
- For any commutative ring A and for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$, the localization $A_\mathfrak{p}$ is a local ring with the maximal ideal $\mathfrak{p}A_\mathfrak{p}$.

Lemma 5.7. (1) *Let A be a local ring. Then the maximal ideal of A coincides with $A \setminus A^\times$.*

(2) *A commutative ring A is a local ring if and only if the set $A \setminus A^\times$ of non-units of A forms an ideal of A.*

PROOF. (1) Assume A is a local ring with the maximal ideal $\mathfrak{m}$. Then for any element $f \in A \setminus A^\times$, an ideal $I = fA + \mathfrak{m}$ is an ideal of A. By Zorn's lemma, we know that I is contained in a maximal ideal of A. From the assumption, the maximal ideal should be $\mathfrak{m}$. Therefore, we have

$$fA \subset \mathfrak{m}$$

which shows that

$$A \setminus A^\times \subset \mathfrak{m}.$$

The converse inclusion being obvious (why?), we have

$$A \setminus A^\times = \mathfrak{m}.$$

(2) The "only if" part is an easy corollary of (1). The "if" part is also easy.

□

COROLLARY 5.8. *Let A be a commutative ring. Let $\mathfrak{p}$ its prime ideal. Then $A_\mathfrak{p}$ is a local ring with the only maximal ideal $\mathfrak{p}A_\mathfrak{p}$.*

DEFINITION 5.9. Let A, B be local rings with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$ respectively. A local homomorphism $\varphi : A \to B$ is a homomorphism which preserves maximal ideals. That means, a homomorphism φ is said to be loc al if

$$\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$$

EXAMPLE 5.10 (of NOT being a local homomorphism).

$$\mathbb{Z}_{(p)} \to \mathbb{Q}$$

is not a local homomorphism.

LEMMA 5.11 (Zorn's lemma). *Let $\mathcal{S}$ be a partially ordered set. Assume that every totally ordered subset of $\mathcal{S}$ has an upper bound in $\mathcal{S}$. Then $\mathcal{S}$ has at least one maximal element.*

PROPOSITION 5.12. *Let A be a commutative ring. let I be an ideal of A such that $A \neq I$. Then there exists a maximal ideal $\mathfrak{m}$ of A which contains I.*

THEOREM 5.13 (Nakayama's lemma, or NAK). *Let A be a commutative ring. Let M be an A-module. We assume that M is finitely generated (as a module) over A. That means, there exists a finite set of elements $\{m_i\}_{i=1}^t$ such that*

$$M = \sum_{i=1}^{t} Am_i$$

holds. If an ideal I of A satisfies

$$IM = M \quad (\text{that is, } M/IM = 0),$$

then there exists an element $c \in I$ such that

$$cm = m \qquad (\forall m \in M)$$

holds. If furthermore I is contained in $\cap_{\mathfrak{m} \in \mathrm{Spm}(A)} \mathfrak{m}$ (the Jacobson radical of A), then we have $M = 0$.

Proof. Since $IM = M$, there exists elements $b_{il} \in I$ such that

$$a_i = \sum_{l=1}^{t} b_{il} a_l \qquad (1 \leq i \leq t)$$

holds. In a matrix notation, this may be rewritten as

$$v = Bv$$

with $v =^t (m_1, \ldots, m_n)$, $B = (b_{ij}) \in M_t(I)$. Using the unit matrix $1_t \in M_t(A)$ one may also write :

$$(1_t - B)v = 0.$$

Now let R be the adjugate matrix of $1_t - B$. In other words, it is a matrix which satisfies

$$R(1_t - B) = (1_t - B)R = (\det(1_t - B))1_t.$$

Then we have

$$\det(1_t - B) \cdot v = R(1_t - B)v = 0.$$

On the other hand, since $1_t - B = 1_t$ modulo I, we have $\det(1_t - B) = 1 - c$ for some $c \in I$. This c clearly satisfies

$$v = cv.$$

□

We need a criterion for regularity. Instead of developing the vast theory of regular rings, we site here the following theorem:

Theorem 5.14. *(Matsumura "commutative ring theory" Theorem14.2) Let $(R, \mathfrak{m})$ be a regular local ring. then the following two statements are equivalent:*

1. *The images in $\mathfrak{m}/\mathfrak{m}^2$ of $x_1, \ldots, x_i$ are linearly independent over $R/\mathfrak{m}$.*
2. *$R/(x_1, \ldots, x_i)$ is an $(n - i)$-dimensional regular local ring.*