ZETA FUNCTIONS. NO.4

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We quote the famous **Weil conjecture**

Conjecture 4.1 (Now a theorem [1]). Let X be a projective smooth variety of dimension d. Then:

W1. (Rationality)

$$Z(X,T) = \frac{P_1(X,T)P_3(X,T)\dots P_{2d-1}(X,T)}{P_0(X,T)P_2(X,T)\dots P_{2d}(X,T)}$$

W2. (Integrality) $P_0(X,T) = 1-T$, $P_{2d}(X,T) = 1-q^dT$, and for each r, P_r is a polynomial in $\mathbb{Z}[T]$ which is factorized as

$$P_r(X,T) = \prod(1-a_{r,i}T)$$

where $a_{r,i}$ are algebraic integers.

W3. (Functional Equation)

$$Z(X,1/q^dT) = \pm q^{d\chi/2}T^\chi Z(t)$$

where $\chi = (\Delta.\Delta)$ is an integer.

W4. (Rieman Hypothesis) each $a_{r,i}$ and its conjugates have absolute value $q^{r/2}$.

W5. If X is the specialization of a smooth projective variety Y over a number field, then the degeee of $P_r(X,T)$ is equal to the r-th Betti number of the complex manifold $Y(\mathbb{C})$. (When this is the case, the number χ above is equal to the "Euler characteristic" $\chi = \sum_i(-1)^i b_i$ of $Y(\mathbb{C})$.)

It is a profound theorem, relating rational points $X(\mathbb{F}_q)$ of X over finite fields and topology of $Y(\mathbb{C})$.

The following proposition (which is a precursor of the above conjecture) is a special case

Proposition 4.2 (Weil). *Let E be an elliptic curve over $\mathbb{F}_q$. Then we have*

$$Z(E/\mathbb{F}_q,T) = \frac{1-aT+qT^2}{(1-T)(1-qT)}.$$

where a is an integer which satisfies $|a| \leq 2\sqrt{q}$.

[1] There are a lot of people who contributed. See the references.

Note that for each E we have only one unknown integer a to determine the Zeta function. So it is enough to compute $\#E(\mathbb{F}_q)$. to compute the Zeta function of E. (When $q = p$ then one may use the result in the preceding section.)

For a further study we recommend [1, Appendix C],[2].

References

[1] R. Hartshorne, *Algebraic geometry*, Springer Verlag, 1977.
[2] J. S. Milne, *Étale cohomology*, Princeton University Press, 1980.