# COMMUTATIVE ALGEBRA

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## 09. Ext, Tor

Let $\mathcal{C}$ be an abelian category. For any object $M$ of $\mathcal{C}$, the extension group $\mathrm{Ext}^j_{\mathcal{C}}(M, N)$ is defined to be the derived functor of the "hom" functor

$$N \mapsto \mathrm{Hom}_{\mathcal{C}}(M, N).$$

We note that the Hom functor is a "bifunctor". We may thus consider the right derived functor of $\bullet \mapsto \mathrm{Hom}(\bullet, N)$ and that of $\bullet \mapsto \mathrm{Hom}(M, \bullet, N)$. Fortunately, both coincide: The extension group $\mathrm{Ext}^\bullet_{\mathcal{C}}(M, N)$ may be calculated by using either an injective resolution of the second variable $N$ or a projective resoltuion of the first variable $M$.

EXAMPLE 9.1. Let us compute the extension groups $\mathrm{Ext}^j_{\mathbb{Z}}(\mathbb{Z}/36\mathbb{Z}, \mathbb{Z}/108\mathbb{Z})$.

(1) We may compute them by using an injective resolution

$$0 \to \mathbb{Z}/108\mathbb{Z} \to \mathbb{Q}/108\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$$

of $\mathbb{Z}/108\mathbb{Z}$.

(2) We may compute them by using a free resolution

$$0 \leftarrow \mathbb{Z}/36\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 36\mathbb{Z} \leftarrow 0$$

of $\mathbb{Z}/36\mathbb{Z}$.

EXERCISE 9.1. Compute an extension group $\mathrm{Ext}^j(M, N)$ for modules $M, N$ of your choice. (Please choose a non-trivial example).

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DEFINITION 9.2. Let $A$ be an associative unital (but not necessarily commutative) ring. Let $L$ be a right $A$-module. Let $M$ be a left $A$-module. For any ($\mathbb{Z}$-)module $N$, an map

$$\varphi : L \times M \to N$$

is called an **$A$-balanced biadditive map** if

(1) $\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \quad (\forall x_1, \forall x_2 \in L, \forall y \in M)$.
(2) $\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2) \quad (\forall x \in L, \forall y_1, \forall y_2 \in M)$.
(3) $\varphi(xa, y) = \varphi(x, ay) \quad (\forall x \in L, \forall y \in M, \forall a \in A)$.

PROPOSITION 9.3. *Let $A$ be an associative unital (but not necessarily commutative) ring. Then for any right $A$-module $L$ and for any left $A$-module $M$, there exists a ($\mathbb{Z}$-)module $X_{L,M}$ together with a $A$-balanced map*

$$\varphi_0 : L \times M \to X_{L,M}$$

*which is universal amoung $A$-balanced maps.*

DEFINITION 9.4. We employ the assumption of the proposition above. By a standard argument on universal objects, we see that such object is unique up to a unique isomorphism. We call it the **tensor product** of $L$ and $M$ and denote it by

$$L \otimes_A M.$$

LEMMA 9.5. *Let $A$ be an associative unital ring. Then:*

(1) $A \otimes_A M \cong M$.
(2) $(L_1 \oplus L_2) \otimes_A M \cong (L_1 \otimes M) \oplus (L_2 \otimes_A M)$.
(3) *For any $M$, $L \mapsto L \otimes_A M$ is a right exact functor.*
(4) *For any right ideal $J$ of $A$ and for any $A$-module $M$, we have*

$$(A/J) \otimes_A M \cong M/J.M$$

*In particular, if the ring $A$ is commutative, then for any ideals $I, J$ of $A$, we have*

$$(A/I) \otimes_A (A/J) \cong A/(I + J)$$

DEFINITION 9.6. For any left $A$-module $M$, the left derived functor $L_j F(M)$ of $F_M = \bullet \otimes_A M$ is called the Tor functor and denoted by $\mathrm{Tor}_j^A(\bullet, M)$.

By definition, $\mathrm{Tor}_j^A(L, M)$ may be computed by using projective resolutions of $L$.

EXERCISE 9.2. Compute $\mathrm{Tor}_j^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for $n, m \in \mathbb{Z}_{>0}$.