COMMUTATIVE ALGEBRA

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08. Homological algebra

DEFINITION 8.1. Let R be a ring. A **cochain complex** of R-modules is a sequence of R-modules

$$C^\bullet : \dots \overset{d^{n-1}}{\to} C^n \overset{d^n}{\to} C^{n+1} \overset{d^{n+1}}{\to} \dots$$

such that $d^n \circ d^{n-1} = 0$. The n-th **cohomology** of the cochain complex is defined to be the R-module

$$H^n(C^\bullet) = \operatorname{Ker}(d^n)/\operatorname{Image}(d^{n-1}).$$

Elements of $\operatorname{Ker}(d^n)$ (respectively, $\operatorname{Image}(d^{n-1})$) are often referred to as **cocycles** (respectively, **coboundaries**).

DEFINITION 8.2. Let R be a ring.

(1) An R-module I is said to be **injective** if it satisfies the following condition: For any R-module homomorphism $f : M \to I$ and for any monic R-module homomorphism $\iota : N \to M$, f "extends" to an R-module homomorphism $\hat{f} : M \to I$.

$$\begin{array}{ccc} M & \xrightarrow{\hat{f}} & I \\ {\scriptstyle\iota}\uparrow & & \| \\ N & \xrightarrow{f} & I \end{array}$$

(2) A R-module P is said to be **projective** if it satisfies the following condition: For any R-module homomorphism $f : P \to N$ and for any epic R-module homomorphism $\pi : M \to N$, f "lifts" to a morphism $\hat{f} : M \to I$.

$$\begin{array}{ccc} P & \xrightarrow{\hat{f}} & M \\ \| & & {\scriptstyle\pi}\downarrow \\ P & \xrightarrow{f} & N \end{array}$$

EXERCISE 8.1. Let R be a ring. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of R-modules. Assume furthermore that M_3 is projective. Then show that the sequence

$$0 \to \operatorname{Hom}_R(N, M_1) \overset{\operatorname{Hom}_R(N,f)}{\to} \operatorname{Hom}_R(N, M_2) \overset{\operatorname{Hom}_R(N,g)}{\to} \operatorname{Hom}_R(N, M_3) \to 0$$

is exact.

LEMMA 8.3. *Let R be a (unital associative but not necessarily commutative) ring. Then for any R-module M, the following conditions are equivalent.*

(1) *M is a direct summand of free modules.*
(2) *M is projective*

Corollary 8.4. *For any ring R, the category* $(R\text{-modules})$ *of R-modules* **have enough projectives.** *That means, for any object $M \in (R\text{-modules})$, there exists a projective object P and a surjective morphism $f : P \to M$.*

Definition 8.5. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \overset{r\times}{\to} M$$

is surjective.

Definition 8.6. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \overset{r\times}{\to} M$$

is epic.

Lemma 8.7. *Let R be a (commutative) principal ideal domain (PID). Then an R-module I is injective if and only if it is divisible.*

Proposition 8.8. *For any (not necessarily commutative) ring R, the category* $(R\text{-modules})$ *of R-modules* **has enough injectives.** *That means, for any object $M \in (R\text{-modules})$, there exists an injective object I and an monic morphism $f : M \to I$.*

A bit of category theory:

Definition 8.9. A **category** $\mathcal{C}$ is a collection of the following data

(1) A collection $\mathrm{Ob}(\mathcal{C})$ of **objects** of $\mathcal{C}$.
(2) For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set
$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$
of **morphisms**.
(3) For each triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map(“composition (rule)”)
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

satisfying the following axioms

(1) $\mathrm{Hom}(X, Y) \cap \mathrm{Hom}(Z, W) = \emptyset$ unless $(X, Y) = (Z, W)$.
(2) (Existence of an identity) For any $X \in \mathrm{Ob}(\mathcal{C})$, there exists an element $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ such that
$$\mathrm{id}_X \circ f = f, \quad g \circ \mathrm{id}_X = g$$
holds for any $f \in \mathrm{Hom}(S, X), g \in \mathrm{Hom}(X, T)$ $(\forall S, T \in \mathrm{Ob}(\mathcal{C}))$.
(3) (Associativity) For any objects $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$, and for any morphisms $f \in \mathrm{Hom}(X, Y), g \in \mathrm{Hom}(Y, Z), h \in \mathrm{Hom}(Z, W)$, we have
$$(f \circ g) \circ h = f \circ (g \circ h).$$

Morphisms are the basic actor/actoress in category theory.

An additive category is a category in which one may “add” some morphisms.

Definition 8.10. An additive category $\mathcal{C}$ is said to be **abelian** if it satisfies the following axioms.

(A4-1) Every morphism $f : X \to Y$ in $\mathcal{C}$ has a kernel $\ker(f) : \mathrm{Ker}(f) \to X$.
(A4-2) Every morphism $f : X \to Y$ in $\mathcal{C}$ has a cokernel $\mathrm{coker}(f) : Y \to \mathrm{Coker}(f)$.
(A4-3) For any given morphism $f : X \to Y$, we have a suitably defined isomorphism

$$l : \mathrm{Coker}(\ker(f)) \cong \mathrm{Ker}(\mathrm{coker}(f))$$

in $\mathcal{C}$. More precisely, l is a morphism which is defined by the following relations:

$$\ker(\mathrm{coker}(f)) \circ \overline{f} = f \ (\exists \overline{f}), \quad \overline{f} = l \circ \mathrm{coker}(\ker(f)).$$

Definition 8.11. A (covariant) **functor** F from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of the following data:

(1) An function which assigns to each object C of $\mathcal{C}$ an object $F(C)$ of $\mathcal{D}$.
(2) An function which assigns to each morphism f of $\mathcal{C}$ an morphism $F(f)$ of $\mathcal{D}$.

The data must satisfy the following axioms:

(functor-1) $F(1_C) = 1_{F(C)}$ for any object C of $\mathcal{C}$.
(functor-2) $F(f \circ g) = F(f) \circ F(g)$ for any composable morphisms f, g of $\mathcal{C}$.

By employing the following axiom instead of the axiom (functor-2) above, we obtain a definition of a **contravariant functor**:
(functor-2′) $F(f \circ g) = F(g) \circ F(f)$ for any composable morphisms

Definition 8.12. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor between additive categories. We call F **additive** if for any objects M, N in $\mathcal{C}_1$,

$$\mathrm{Hom}(M, N) \to \mathrm{Hom}(F(M), F(N))$$

is additive.

Definition 8.13. Let F be an additive functor from an abelian category $\mathcal{C}_1$ to $\mathcal{C}_2$.

(1) F is said to be **left exact** (respectively, **right exact**) if for any exact sequence

$$0 \to L \to M \to N \to 0,$$

the corresponding map

$$0 \to F(L) \to F(M) \to F(N)$$

(respectively,

$$F(L) \to F(M) \to F(N) \to 0)$$

is exact
(2) F is said to be **exact** if it is both left exact and right exact.

Lemma 8.14. *Let R be a (unital associative but not necessarily commutative) ring. Then for any R-module M, the following conditions are equivalent.*

(1) *M is a direct summand of free modules.*
(2) *M is projective*

Corollary 8.15. *For any ring R, the category* (R-modules) *of R-modules* **have enough projectives.** *That means, for any object $M \in$* (R-modules)*, there exists a projective object P and a surjective morphism $f : P \to M$.*

Definition 8.16. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \stackrel{r\times}{\to} M$$

is surjective.

Definition 8.17. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \stackrel{r\times}{\to} M$$

is epic.

Definition 8.18. Let $(K^\bullet, d_K)$, $(L^\bullet, d_L)$ be complexes of objects of an additive category $\mathcal{C}$.

(1) A **morphism of complex** $u : K^\bullet \to L^\bullet$ is a family

$$u^j : K^j \to L^j$$

of morphisms in $\mathcal{C}$ such that u commutes with d. That means,

$$u^{j+1} \circ d_K^j = d_K^j \circ u^j$$

holds.

(2) A **homotopy** between two morphisms $u, v : K^\bullet \to L\bullet$ of complexes is a family of morphisms

$$h^j : K^j \to L^{j-1}$$

such that $u - v = d \circ h + h \circ d$ holds.

Lemma 8.19. *Let $\mathcal{C}$ be an abelian category that has enough injectives. Then:*

(1) *For any object M in $\mathcal{C}$, there exists an* **injective resolution** *of M. That means, there exists an complex $I^\bullet$ and a morphism $\iota_M : M \to I^0$ such that*

$$H^j(I^\bullet) = \begin{cases} M \ (\text{via } \iota_M) & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}$$

(2) *For any morphism $f : M \to N$ of $\mathcal{C}$, and for any injective resolutions $(I^\bullet, \iota_M)$, $(J^\bullet, \iota_N)$ of M and N (respectively), There exists a morphism $\bar{f} : I^\bullet \to J^\bullet$ of complexes which commutes with f. Forthermore, if there are two such morphisms $\bar{f}$ and f', then the two are homotopic.*

Definition 8.20. Let $\mathcal{C}_1$ be an abelian category which has enough injectives. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a left exact functor to an abelian category. Then for any object M of $\mathcal{C}_1$ we take an injective resolution $I_M^\bullet$ of M and define

$$R^i F(M) = H^i(I_M^\bullet).$$

and call it the derived functor of F.

Lemma 8.21. *The derived functor is indeed a functor.*