COMMUTATIVE ALGEBRA

YOSHIFUMI TSUCHIMOTO

04. Dimension

Recall that for any commutative ring A, we define its (Krull) dimension $\dim(A)$ as the Krull dimension of $\operatorname{Spec}(A)$.

Definition 4.1. Let A be a commutative ring. For any A-module M, we define its dimension as

$$\dim(M) = \dim(A/\operatorname{Ann}(M)).$$

where $\operatorname{Ann}(M) = \{x \in A; x.M = 0\}$.

Definition 4.2. For any A-module M of a ring A, we define its length $l(M)$ as the supremum of the lenths of descending chains of submodules of M.

Example 4.3. Let k be a commutative field. A k-module V is a vector space over k. The lengh of V is then equal to the dimension of V as a k-vector space. In what follows, we denote it as $\operatorname{rank}_k(V)$.

Exercise 4.1. Compute the lenth of a $\mathbb{Z}$-module $\mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.

Definition 4.4. Let $A = (A, \mathfrak{m})$ be a local ring. Let M be an A-module. we define $\delta(A)$ to be the smallest value of n such that there exist $x_1, x_2, \dots, x_n \in \mathfrak{m}$ for which $l(M/x_1M + \dots + x_nM) < \infty$.

Let us recall the definition of Noetherian ring.

Definition 4.5. A ring is called **Noetherian** if any asscending chain

$$I_1 \subset I_2 \subset I_3 \subset \dots$$

stops after a finite number of steps. (That means, There exists a number N such that $I_N = I_{N+1} = I_{N+2} = \dots$.)

Proposition 4.6. *A commutative ring A is Noetherina if and only if its ideals are always finitely generated.*

Definition 4.7. Let $(A, \mathfrak{m})$ be a be a Noetherian local ring. Let I be an ideal of A. We say that I is an ideal of definition if there exists an integer j such that $I \supset \mathfrak{m}^j$. Then for any finite A-module M, we define

$$\chi_M^I(n) = l(M/I^{n+1}M).$$

It is known that there exists a polynomal p_M^I such that $\chi_M^I(n) = p_M^I(n)$ for $n >> 0$. We define $d(M)$ as the degree of the polynomial p. d does not depend on the choice of the ideal I of definition.

Proposition 4.8. *For any Noetherian local ring A and for any finite A-module M, we have*

$$d(M) = \dim(M) = \delta(M).$$

Definition 4.9. For any local ring A, we define its embedding dimension as $\operatorname{rank}_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$.

Definition 4.10. A Noetherian local ring is said to be regular if its embedding dimension is equal to the dimension of A.