COMMUTATIVE ALGEBRA

YOSHIFUMI TSUCHIMOTO

06. Integral elements, normal closure

Definition 6.1. Let S be a ring which contains a subring R. An element s of S is said to be **integral** over R if it is a root of a monic polynomial in $R[X]$.

Lemma 6.2. *Let S be a ring which contains a subring R. For any element s of S, the following conditions are equivalent:*

(1) *s is integral over R.*
(2) *$R[s]$ is a finite R-module.*
(3) *There exists a subring S_1 of S which contains $R[s]$ as a subset such that S_1 is a finite module over R.*

Proposition 6.3. *Let S be a ring which contains a subring R. Then the set of all elements of S which are integral over R is a subring of S. (We call it the* **integral closure** *of R in S.)*

Example 6.4. Each element of $\mathbb{C}$ which is integral over $\mathbb{Z}$ is said to be an **algebraic integer**. The set of algebraic integers forms a subring of $\mathbb{C}$.

Definition 6.5. Let R be an integral domain. Let us denote its field of quotients by $Q(R)$. The integral closure of R in $Q(R)$ is called the **normalization** of R. R is called **normal** if it is equal to its normalizaiton.

By using the Gauss's lemma, we see that every PID is normal.

Normalizations are useful to reduce singularities.

Example 6.6. Let us put

$$R = \mathbb{C}[X, Y]/(Y^2 - X^2(X + 1))$$

and denote the class of X, Y in R by x, y respectively. R is not normal. Indeed, $z = y/x$ satisfies a monic equation

$$z^2 - (x + 1) = 0.$$

Thus the normalization $\bar{R}$ of R contains the element z. Now, let us note that equation

$$x = z^2 - 1, \quad y = zx = z(z^2 - 1)$$

holds so that $R[z] = \mathbb{C}[z]$ holds. Since $\mathbb{C}[z]$ is normal, we see that $\bar{R} = R[z] = \mathbb{C}[z]$. Note that $\Omega^1_{R/\mathbb{C}}$ is not locally free whereas $\Omega^1_{\bar{R}/\mathbb{C}}$ is free.

Example 6.7. Let us consider a ring $R = \mathbb{Z}[X]/(u(X))$ where u is a monic element in $\mathbb{Z}[X]$. Let us denote by α the residue class of X in R.

$$\Omega^1_{R/\mathbb{Z}} = RdX/u'(X)RdX \cong R/(u'(\alpha)).$$

EXERCISE 6.1. The normalization of $R = \mathbb{Z}[\sqrt{-3}]$ is equal to $\bar{R} = \mathbb{Z}[\sqrt{-3}]$. Compute $\Omega^1_{R/\mathbb{Z}}$ and $\Omega^1_{\bar{R}/\mathbb{Z}}$.

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THEOREM 6.8 (Matsumura, Corollary of Theorem 23.9). *Let A, B are Noetherian local ring Let $\varphi : A \to B$ is be a local homormophism. If φ is flat morphism, and if B is normal, then A is also normal.*

In other words, a normalization of a ring A can never be flat (unless the trivial case where A itself is normal).

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DEFINITION 6.9. For any commutative ring A, we define its **spectrum** as

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subset A; \mathfrak{p} \text{ is a prime ideal of } A.$$

For any subset S of A we define

$$V(S) = V_{\mathrm{Spec}\, A}(S) = \{\mathfrak{p} \in \mathrm{Spec}\, A; \mathfrak{p} \subset S\}$$

Then we may topologize $\mathrm{Spec}(A)$ in such a way that the closed sets are sets of the form $V(S)$ for some S. Namely,

$$F : \text{closed} \iff \exists S \subset A (F = V(S))$$

We refer to the topology as the **Zariski topology**.

LEMMA 6.10. *For any ring A, the following facts holds.*

1. *For any subset S of A, we have*
$$V(S) = \bigcap_{s \in S} V(\{s\}).$$
2. *For any subset S of A, let us denote by $\langle S \rangle$ the ideal of A generated by S. then we have*
$$V(S) = V(\langle S \rangle)$$

PROPOSITION 6.11. *For any ring homomorphism $\varphi : A \to B$, we have a map*

$$\mathrm{Spec}(\varphi) : \mathrm{Spec}(B) \ni \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p}) \in \mathrm{Spec}(A).$$

It is continuous with respect to the Zariski topology.

PROPOSITION 6.12. *For any ring A, the following statements hold.*

1. *For any ideal I of A, let us denote by $\pi_I : A \to A/I$ the canonical projection. Then $\mathrm{Spec}(\pi_I)$ gives a homeomorphism between $\mathrm{Spec}(A/I)$ and $V_{\mathrm{Spec}\, A}(I)$.*
2. *For any element s of A, let us denote by $\iota_s : A \to A[s^{-1}]$ be the canonical map. Then $\mathrm{Spec}(\iota_s)$ gives a homeomorphism between $\mathrm{Spec}(A[s^{-1}])$ and $\complement V_{\mathrm{Spec}_A}(\{s\})$.*

PROPOSITION 6.13. *Let A, B be a ring. Let $\varphi : A \to B$ be a ring homomoprhism. We regard B as an A module via φ. If B is a finite A-module, then*

$$\mathrm{Spec}(\varphi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is a closed map with respect to the Zariski topology.