COMMUTATIVE ALGEBRA

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05. Formal smoothness, unramifiedness.

Definition 5.1. Let A be a ring. Let I be an ideal of A. The I-adic topology on A is a topology defined by introducing for each $a \in A$, $\{a + I^n\}_{n=1}^{\infty}$ as the neighbourhood base.

It is easy to see that the I-adic topology is given by a quasi-metric defined by

$$d(a, b) = \inf\{\frac{1}{2^n};\ a - b \in I^n\}$$

Proposition 5.2. *Let A be a ring. Let I be an ideal of A. We equip A with the I-adic topology. Then A is Hausdorff if and only if*

$$\bigcap_n I^n = 0.$$

If this is the case, the completion of A is equal to $\varprojlim A/I^n$. Thus A is complete Hausdorff if and only if a cannonically defined map

$$A \to \varprojlim A/I^n$$

is an isomorphism.

Example 5.3. Let p be a prime number. The ring $\mathbb{Z}$ of rational integers equipped with the $p\mathbb{Z}$-adic topology is Hausdorff. Its completion is denoted by $\mathbb{Z}_p$ and is called the ring of p-adic integers.

Definition 5.4. Let A be a ring. Let B be an A-algebra. Let I be an ideal of B. We equip B with the I-adic topology. B is **I-smooth** over A if for any A-algebra C, any ideal N of C satisfying $N^2 = 0$ and any A-algebra homomorphism $u : B \to C/N$ which is continuous with respect to the discrete topology of C/N, there exists a lifting $v : B \to C$ of u to C, as an A-algebra homormophism.

Definition 5.5. Let A be a ring. Let B be an A-algebra. Let I be an ideal of B. We equip B with the I-adic topology. A-algebra B is **I-unramified** over A if for any A-algebra C, any ideal N of C satisfying $N^2 = 0$ and any A-algebra homomorphism $u : B \to C/N$ which is continuous with respect to the discrete topology of C/N, there is at most one lifting $v : B \to C$ of u to C, as an A-algebra homormophism.

Definition 5.6. An A-algebra B is I-étale over A if it is both I-smooth and I-unramified.

Note that the conditions I-smooth/unramified/étale become weaker if we take I larger.

In the "strongest" case where $I = 0$, the continuity of the homomorphism u is automatic (any homomorphism is continuous.) 0-smoothness (respectively, 0-unramifiedness, respectively, 0-étale-ness) is also refered to as formal smoothness (respectively, formal unramifiedness, respectively, formal étale-ness).