COMMUTATIVE ALGEBRA

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03. Derivations and differentials

DEFINITION 3.1. Let A be a ring. A ring B is called an A-**algebra** if there is given a distinguished ring homomorphism (which is called the structure morphism) from A to B.

DEFINITION 3.2. Let A be a commutative ring. Let B be an A-algebra. Let M be an B-module. A map $D : A \to M$ is called an A-derivation if it satisfies the following conditions.

(1) D is A-linear.
(2) $D(xy) = xD(y) + D(x)y \qquad (\forall x, \forall y \in B)$.

PROPOSITION 3.3. *For any algebra B over a ring A, There exists an universal derivation $d : B \to \Omega^1_{B/A}$.*

DEFINITION 3.4. The module $\Omega^1_{B/A}$ is called the module of differentials of B over A.

DEFINITION 3.5. An algebra B over a ring A is called **unramified** over A if $\Omega^1_{B/A} = 0$. B is called **étale** over A if it is unramified and flat.

LEMMA 3.6. *Let A be a commutative ring. Let S be its multiplicative subset. Then $S^{-1}A$ is étale over A.*