COHOMOLOGIES.

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07. Tensor products and Tor

DEFINITION 7.1. Let A be an associative unital (but not necessarily commutative) ring. Let L be a right A-module. Let M be a left A-module. For any ($\mathbb{Z}$-)module N, an map

$$\varphi : L \times M \to N$$

is called an **A-balanced biadditive map** if

(1) $\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \quad (\forall x_1, \forall x_2 \in L, \forall y \in M)$.
(2) $\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2) \quad (\forall x \in L, \forall y_1, \forall y_2 \in M)$.
(3) $\varphi(xa, y) = \varphi(x, ay) \quad (\forall x \in L, \forall y \in M, \forall a \in A)$.

PROPOSITION 7.2. *Let A be an associative unital (but not necessarily commutative) ring. Then for any right A-module L and for any left A-module M, there exists a ($\mathbb{Z}$-)module $X_{L,M}$ together with a A-balanced map*

$$\varphi_0 : L \times M \to X_{L,M}$$

which is universal amoung A-balanced maps.

DEFINITION 7.3. We employ the assumption of the proposition above. By a standard argument on universal objects, we see that such object is unique up to a unique isomorphism. We call it the **tensor product** of L and M and denote it by

$$L \otimes_A M.$$

LEMMA 7.4. *Let A be an associative unital ring. Then:*

(1) $A \otimes_A M \cong M$.
(2) $(L_1 \oplus L_2) \otimes_A M \cong (L_1 \otimes M) \oplus (L_2 \otimes_A M)$.
(3) *For any M, $L \mapsto L \otimes_A M$ is a right exact functor.*
(4) *For any right ideal J of A and for any A-module M, we have*

$$(A/J) \otimes_A M \cong M/J.M$$

In particular, if the ring A is commutative, then for any ideals I, J of A, we have

$$(A/I) \otimes_A (A/J) \cong A/(I + J)$$

DEFINITION 7.5. For any left A-module M, the left derived functor $L_j F(M)$ of $F_M = \bullet \otimes_A M$ is called the Tor functor and denoted by $\mathrm{Tor}_j^A(\bullet, M)$.

By definition, $\mathrm{Tor}_j^A(L, M)$ may be computed by using projective resolutions of L.

EXERCISE 7.1. Compute $\mathrm{Tor}_j^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for $n, m \in \mathbb{Z}_{>0}$.