COHOMOLOGIES.

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06. Ext as a derived functor

Let $\mathcal{C}$ be an abelian category. For any object M of $\mathcal{C}$, the extension group $\mathrm{Ext}^j_{\mathcal{C}}(M,N)$ is defined to be the derived functor of the "hom" functor

$$N \mapsto \mathrm{Hom}_{\mathcal{C}}(M,N).$$

We note that the Hom functor is a "bifunctor". We may thus consider the right derived functor of $\bullet \mapsto \mathrm{Hom}(\bullet,N)$ and that of $\bullet \mapsto \mathrm{Hom}(M,\bullet,N)$. Fortunately, both coincide: The extension group $\mathrm{Ext}^\bullet_{\mathcal{C}}(M,N)$ may be calculated by using either an injective resolution of the second variable N or a projective resoltuion of the first variable M. See [1, Proposition 8.4,Corollary 8.5].

EXAMPLE 6.1. Let us compute the extension groups $\mathrm{Ext}^j_{\mathbb{Z}}(\mathbb{Z}/36\mathbb{Z},\mathbb{Z}/108\mathbb{Z})$.

(1) We may compute them by using an injective resolution
$$0 \to \mathbb{Z}/108\mathbb{Z} \to \mathbb{Q}/108\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$$
of $\mathbb{Z}/108\mathbb{Z}$.

(2) We may compute them by using a free resolution
$$0 \leftarrow \mathbb{Z}/36\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 36\mathbb{Z} \leftarrow 0$$
of $\mathbb{Z}/36\mathbb{Z}$.

EXERCISE 6.1. Compute an extension group $\mathrm{Ext}^j(M,N)$ for modules M,N of your choice. (Please choose a non-trivial example).

In the last lecture we mentioned the notion of injective hulls. Although they are not essential part of our lecture, students may find it interesting to calculate some of the injective hulls of known modules. So we write down some definitions and results related to them.

DEFINITION 6.2. Let M be an R-module. An R-module $E \supset M$ is called an **essential extension** of M if every non-zero submodule of E intersect M non-trivially. We denote this as $E \supset_e M$.

Such an essential extension is called maximal if no module properly containing E is an essential extension of M .

LEMMA 6.3. *A module M is injective if and only if M has no proper essential extensions.*

LEMMA 6.4. *Let R be a ring. Let $F \subset M$ be R-modules. We consider a family $\mathcal{F}$ of modules E which satisfy the following properties.*

- *E is an R-submodule of F which contains M.*
- *E is an essential extension of M.*

Then:

(1) *The set $\mathcal{F}$ has a maximal element.*
(2) *If F is an injective R-module, then any maximal element E of $\mathcal{F}$ is injective.*

THEOREM 6.5. *For any R-module M, there exists an injective module I which contains M whichis minimal among such. The module I is unique up to a (non-unique) isomorphism.*

DEFINITION 6.6. Such I in the above theorem is called the **injective hull** of M.

Injective hulls may then be used to obtain the "minimal injective resolution" of a module.

EXAMPLE 6.7. Let n be a positive integer. The injective hull of a $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is equal to $\mathbb{Z}[\frac{1}{n}]/n\mathbb{Z}$. Thus an injective resolution of $\mathbb{Z}/n\mathbb{Z}$ is given as follows.

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}[\frac{1}{n}]/n\mathbb{Z} \to \mathbb{Z}[\frac{1}{n}]/\mathbb{Z} \to 0$$

REFERENCES

[1] S. Lang, *Algebra (graduate texts in mathematics)*, Springer Verlag, 2002.