COHOMOLOGIES.

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05. projective and injective modules

DEFINITION 5.1. A (covariant) **functor** F from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of the following data:

(1) An function which assigns to each object C of $\mathcal{C}$ an object $F(C)$ of $\mathcal{D}$.
(2) An function which assigns to each morphism f of $\mathcal{C}$ an morphism $F(f)$ of $\mathcal{D}$.

The data must satisfy the following axioms:

(functor-1) $F(1_C) = 1_{F(C)}$ for any object C of $\mathcal{C}$.
(functor-2) $F(f \circ g) = F(f) \circ F(g)$ for any composable morphisms f, g of $\mathcal{C}$.

By employing the following axiom instead of the axiom (functor-2) above, we obtain a definition of a **contravariant functor**:
(functor-2′) $F(f \circ g) = F(g) \circ F(f)$ for any composable morphisms

DEFINITION 5.2. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor between additive categories. We call F **additive** if for any objects M, N in $\mathcal{C}_1$,

$$\operatorname{Hom}(M, N) \to \operatorname{Hom}(F(M), F(N))$$

is additive.

DEFINITION 5.3. Let F be an additive functor from an abelian category $\mathcal{C}_1$ to $\mathcal{C}_2$.

(1) F is said to be **left exact** (respectively, **right exact**) if for any exact sequence

$$0 \to L \to M \to N \to 0,$$

the corresponding map

$$0 \to F(L) \to F(M) \to F(N)$$

(respectively,

$$F(L) \to F(M) \to F(N) \to 0)$$

is exact

(2) F is said to be **exact** if it is both left exact and right exact.

LEMMA 5.4. *Let R be a (unital associative but not necessarily commutative) ring. Then for any R-module M, the following conditions are equivalent.*

(1) *M is a direct summand of free modules.*
(2) *M is projective*

COROLLARY 5.5. *For any ring R, the category* (R-modules) *of R-modules* **have enough projectives.** *That means, for any object $M \in$* (R-modules)*, there exists a projective object P and a surjective morphism $f : P \to M$.*

Definition 5.6. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \overset{r\times}{\to} M$$

is surjective.

Definition 5.7. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \overset{r\times}{\to} M$$

is epic.

Definition 5.8. Let $(K^\bullet, d_K)$, $(L^\bullet, d_L)$ be complexes of objects of an additive category $\mathcal{C}$.

(1) A **morphism of complex** $u : K^\bullet \to L^\bullet$ is a family

$$u^j : K^j \to L^j$$

of morphisms in $\mathcal{C}$ such that u commutes with d. That means,

$$u^{j+1} \circ d_K^j = d_K^j \circ u^j$$

holds.

(2) A **homotopy** between two morphisms $u, v : K^\bullet \to L\bullet$ of complexes is a family of morphisms

$$h^j : K^j \to L^{j-1}$$

such that $u - v = d \circ h + h \circ d$ holds.

Lemma 5.9. *Let $\mathcal{C}$ be an abelian category that has enough injectives. Then:*

(1) *For any object M in $\mathcal{C}$, there exists an* **injective resolution** *of M. That means, there exists an complex $I^\bullet$ and a morphism $\iota_M : M \to I^0$ such that*

$$H^j(I^\bullet) = \begin{cases} M \ (\text{via } \iota_M) & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}$$

(2) *For any morphism $f : M \to N$ of $\mathcal{C}$, and for any injective resolutions $(I^\bullet, \iota_M)$, $(J^\bullet, \iota_N)$ of M and N (respectively), There exists a morphism $\bar{f} : I^\bullet \to J^\bullet$ of complexes which commutes with f. Forthermore, if there are two such morphisms $\bar{f}$ and f', then the two are homotopic.*

Definition 5.10. Let $\mathcal{C}_1$ be an abelian category which has enough injectives. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a left exact functor to an abelian category. Then for any object M of $\mathcal{C}_1$ we take an injective resolution $I_M^\bullet$ of M and define

$$R^i F(M) = H^i(I_M^\bullet).$$

and call it the derived functor of F.

Lemma 5.11. *The derived functor is indeed a functor.*