COHOMOLOGIES.

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03. cohomology of a complex. We mainly follow the treatment in [1].

Definition 3.1. Let R be a ring. A **cochain complex** of R-modules is a sequence of R-modules

$$C^\bullet : \dots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \dots$$

such that $d^n \circ d^{n-1} = 0$. The n-th **cohomology** of the cochain complex is defined to be the R-module

$$H^n(C^\bullet) = \operatorname{Ker}(d^n)/\operatorname{Image}(d^{n-1}).$$

Elements of $\operatorname{Ker}(d^n)$ (respectively, $\operatorname{Image}(d^{n-1})$) are often referred to as **cocycles** (respectively, **coboundaries**).

A bit of category theory:

Definition 3.2. A **category** $\mathcal{C}$ is a collection of the following data

(1) A collection $\operatorname{Ob}(\mathcal{C})$ of **objects** of $\mathcal{C}$.
(2) For each pair of objects $X, Y \in \operatorname{Ob}(\mathcal{C})$, a set

$$\operatorname{Hom}_{\mathcal{C}}(X, Y)$$

of **morphisms**.
(3) For each triple of objects $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$, a map("composition (rule)")

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) \times \operatorname{Hom}_{\mathcal{C}}(Y, Z) \to \operatorname{Hom}_{\mathcal{C}}(X, Z)$$

satisfying the following axioms

(1) $\operatorname{Hom}(X, Y) \cap \operatorname{Hom}(Z, W) = \emptyset$ unless $(X, Y) = (Z, W)$.
(2) (Existence of an identity) For any $X \in \operatorname{Ob}(\mathcal{C})$, there exists an element $\mathrm{id}_X \in \operatorname{Hom}(X, X)$ such that

$$\mathrm{id}_X \circ f = f, \quad g \circ \mathrm{id}_X = g$$

holds for any $f \in \operatorname{Hom}(S, X), g \in \operatorname{Hom}(X, T)$ ($\forall S, T \in \operatorname{Ob}(\mathcal{C})$).
(3) (Associativity) For any objects $X, Y, Z, W \in \operatorname{Ob}(\mathcal{C})$, and for any morphisms $f \in \operatorname{Hom}(X, Y), g \in \operatorname{Hom}(Y, Z), h \in \operatorname{Hom}(Z, W)$, we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Morphisms are the basic actor/actoress in category theory.
An additive category is a category in which one may "add" some morphisms.

Definition 3.3. An additive category $\mathcal{C}$ is said to be **abelian** if it satisfies the following axioms.

(A4-1) Every morphism $f : X \to Y$ in $\mathcal{C}$ has a kernel $\ker(f) : \operatorname{Ker}(f) \to X$.
(A4-2) Every morphism $f : X \to Y$ in $\mathcal{C}$ has a cokernel $\operatorname{coker}(f) : Y \to \operatorname{Coker}(f)$.

(A4-3) For any given morphism $f : X \to Y$, we have a suitably defined isomorphism

$$l : \mathrm{Coker}(\ker(f)) \cong \mathrm{Ker}(\mathrm{coker}(f))$$

in $\mathcal{C}$. More precisely, l is a morphism which is defined by the following relations:

$$\ker(\mathrm{coker}(f)) \circ \overline{f} = f \ (\exists \overline{f}), \quad \overline{f} = l \circ \mathrm{coker}(\ker(f)).$$

DEFINITION 3.4. Let $\mathcal{C}$ be an abelian category.

(1) An object I in $\mathcal{C}$ is said to be **injective** if it satisfies the following condition: For any morphism $f : M \to I$ and for any monic morphism $\iota : N \to M$, f "extends" to a morphism $\hat{f} : M \to I$.

$$\begin{array}{ccc} M & \xrightarrow{\hat{f}} & I \\ {\scriptstyle \iota}\uparrow & & \| \\ N & \xrightarrow{f} & I \end{array}$$

(2) An object P in $\mathcal{C}$ is said to be **projective** if it satisfies the following condition: For any morphism $f : P \to N$ and for any epic morphism $\pi : M \to N$, f "lifts" to a morphism $\hat{f} : M \to I$.

$$\begin{array}{ccc} P & \xrightarrow{\hat{f}} & M \\ \| & & {\scriptstyle \pi}\downarrow \\ P & \xrightarrow{f} & N \end{array}$$

EXERCISE 3.1. Let R be a ring. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of R-modules. Assume furthermore that M_3 is projective. Then show that the sequence

$$0 \to \mathrm{Hom}_R(N, M_1) \overset{\mathrm{Hom}_R(N,f)}{\to} \mathrm{Hom}_R(N, M_2) \overset{\mathrm{Hom}_R(N,g)}{\to} \mathrm{Hom}_R(N, M_3) \to 0$$

is exact.

REFERENCES

[1] S. Lang, *Algebra (graduate texts in mathematics)*, Springer Verlag, 2002.