COHOMOLOGIES.

YOSHIFUMI TSUCHIMOTO

02. "Hom" modules.

LEMMA 2.1. *Let R be a ring. Let $f : M \to N$ be a homomorphism of R-modules. Then for any R-module L we may define:*

(1) *A homomorphism* $\mathrm{Hom}_R(L, f) : \mathrm{Hom}_R(L, M) \to \mathrm{Hom}_R(L, N)$ *defined by* $\mathrm{Hom}_R(L, f)(g) = f \circ g$.
(2) *A homomorphism* $\mathrm{Hom}_R(f, L) : \mathrm{Hom}_R(N, L) \to \mathrm{Hom}_R(M, L)$ *defined by* $\mathrm{Hom}_R(f, L)(h) = h \circ f$.

PROPOSITION 2.2. *Let R be a ring. Let*

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

be an exact sequence of R-modules. Then for any R-module N, we have:

(1)

$$0 \to \mathrm{Hom}_R(N, M_1) \overset{\mathrm{Hom}_R(N,f)}{\to} \mathrm{Hom}_R(N, M_2) \overset{\mathrm{Hom}_R(N,g)}{\to} \mathrm{Hom}_R(N, M_3)$$

is exact. The third arrow $\mathrm{Hom}_R(N, g)$ *need not be surjective.*

(2)

$$0 \to \mathrm{Hom}_R(M_3, N) \overset{\mathrm{Hom}_R(g,N)}{\to} \mathrm{Hom}_R(M_2, N) \overset{\mathrm{Hom}_R(f,N)}{\to} \mathrm{Hom}_R(M_1, N)$$

is exact. The third arrow $\mathrm{Hom}_R(f, N)$ *need not be surjective.*

EXERCISE 2.1. We consider an exact sequence

$$0 \to 3\mathbb{Z} \xrightarrow{i} \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$$

where i is the inclusion map. Show that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \overset{\mathrm{Hom}(i,\mathbb{Z})}{\to} \mathrm{Hom}_{\mathbb{Z}}(3\mathbb{Z}, \mathbb{Z})$$

is not surjective

EXERCISE 2.2. Assume R is a field. Then show that the third arrow which appear in the sequence (1) in Proposition 2.2 is surjective.