COHOMOLOGIES.

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01.Review of elementary definitions on modules.

Definition 1.1. A (unital associative) **ring** is a set R equipped with two binary operations (addition ("+") and multiplication ("·")) such that the following axioms are satisfied.

(Ring1) R is an additive group with respect to the addition.
(Ring2) distributive law holds. Namely, we have

$$a(b+c) = ab + bc, \quad (a+b)c = ac + bc \qquad (\forall a, \forall b, \forall c \in R).$$

(Ring3) The multiplcation is associative.
(Ring4) R has a multiplicative unit.

For any ring R, we denote by 0_R (respectively, 1_R) the zero element of R (respectively, the unit element of R). Namely, 0_R and 1_R are elements of R characterized by the following rules.

- $a + 0_R = a, \quad 0_R + a = a \ \forall a \in R.$
- $a \cdot 1_R = a, \quad 1_R \cdot a = a \ \forall a \in R.$

When no confusion arises, we omit the subscript '$_R$' and write $0, 1$ instead of $0_R, 1_R$.

Definition 1.2. Let R be a unital associative ring. An R**-module** M is an additive group M with R-action

$$R \times M \to M$$

which satisfies

(Mod1) $(r_1 r_2).m = r_1.(r_2.m) \quad (\forall r_1, \forall r_2 \in R, \forall m \in M)$
(Mod2) $1.m = m \quad (\forall m \in M)$
(Mod3) $(r_1 + r_2).m = r_1.m + r_2.m \quad (\forall r_1, \forall r_2 \in R, \forall m \in M).$
(Mod4) $r.(m_1 + m_2) = r.m_1 + r.m_2 \quad (\forall r \in R, \forall m_1, \forall m_2 \in M).$

Example 1.3. Let us give some examples of R-modules.

(1) If k is a field, then the concepts "k-vector space" and "k-module" are identical.
(2) Every abelian group is a module over the ring of integers $\mathbb{Z}$ in a unique way.

Definition 1.4. An subset M of an R-module N is said to be an R**-submodule** of N if M itself is an R-module and the inclusion map $j : M \to N$ is an R-module homomorphism.

Definition 1.5. Let M, N be modules over a ring R. Then a map $f : M \to N$ is called an R**-module homomorphism** if it is additive and preserves the R-action.

The set of all module homomorphisms from M to N is denoted by $\mathrm{Hom}_R(M, N)$. It has an structure of an module in an obvious manner.

Definition 1.6. An subset N of an R-module M is said to be an R**-submodule** of M if N itself is an R-module and the inclusion map $j : N \to M$ is an R-module homomorphism.

DEFINITION 1.7. Let R be a ring. Let N be an R-submodule of an R-module M. Then we may define the **quotient** M/N by

$$M/N = M/\sim_N$$

where the equivalence relation $\sim_N$ is defined as follows:

$$m_1 \sim_N m_2 \quad \iff \quad m_1 - m_2 \in N.$$

It may be shown that the quotient M/N so defined is actually an R-module and that the natural projection

$$\pi : M \to M/N$$

is an R-module homomorphism.

DEFINITION 1.8. Let $f : M \to N$ be an R-module homomorphism between R-modules. Then we define its **kernel** as follows.

$$\mathrm{Ker}(f) = f^{-1}(0) = \{m \in M; f(m) = 0\}.$$

The kernel and the image of an R-module homomorphism f are R-modules.

THEOREM 1.9. *Let $f : M \to N$ be an R-module homomorphism between R-modules. Then*

$$M/\mathrm{Ker}(f) \cong f(N).$$

DEFINITION 1.10. Let R be a ring. An "sequence"

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

is said to be **an exact sequence of R-modules** if the following conditions are satisfied

(Exact1) M_1, M_2 are R-modules.
(Exact2) f, g are R-module homomorphisms.
(Exact3) $\mathrm{Ker}(g) = \mathrm{Image}(f)$.

For any R-submodule N of an R-module M, we have the following exact sequence.

$$0 \to N \to M \to M/N \to 0$$

EXERCISE 1.1. Compute the following modules.

(1) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z})$.
(2) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$.
(3) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/5\mathbb{Z})$.