CATEGORIES, ABELIAN CATEGORIES AND COHOMOLOGIES.

YOSHIFUMI TSUCHIMOTO

Derived categories

We refer to [1] for a good guide to the theory.

Main idea: Instead of dealing with an object of an additive category $\mathcal{C}$, we deal with complexes of $\mathcal{C}$. But:

(1) We want to regard quasi-isomorphic complexes as the "same".
(2) We want to identify two morphisms to be the same if they are homotopic.

11.1. **Cone of a complex.** Assume we are talking about complexes of objects in an additive category $\mathcal{C}$.

Definition 11.1. [1, 4.1] For any complex $X^\bullet$, we define $TX^\bullet$ to be a complex defined by

$$(TX)^i = X^{i+1}, \quad d_{TX} = -d_X.$$

Definition 11.2. [1, 4.3] Let $u : X^\bullet \to Y^\bullet$ be a morphism of complexes. **The cone** $C_u^\bullet$ of u is defined to be a graded object

$$Y^\bullet \oplus TX^\bullet$$

equipped with the following differential:

$$d\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} d_Y & u \\ 0 & -d_X \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix}$$

Idea 1: Instead of considering kernel and cokernel of a morphism u, we consider its cone C_u.

For any u, we have morphisms (triangle):

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{\iota_Y} C_u^\bullet \xrightarrow{p_{TX}} TX^\bullet.$$

Let us call such a triangle **standard**. Now if $\mathcal{C}$ is abelian, then for each standard triangle as above we have the following long exact sequence:

$$\dots \to H^k(X^\bullet) \to H^k(Y^\bullet) \to H^k(C_u^\bullet) \to H^{k+1}(X^\bullet) \to \dots$$

11.2. **The category** $K(\mathcal{C})$**.**

Definition 11.3. [1, 5.1] For any additive category $\mathcal{C}$, we define $K(\mathcal{C})$ to be

(1) $\mathrm{Ob}(K(\mathcal{C})) = \mathrm{Ob}(C(\mathcal{C}))$ (that means, objects of $K(\mathcal{C})$ are complexes).
(2) For any objects $X^\bullet, Y^\bullet$ of $K(\mathcal{C})$, we define

$$\mathrm{Hom}_{K(\mathcal{C})}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{C(\mathcal{C})}(X^\bullet, Y^\bullet)/\mathrm{Homotopy}$$

Even if $\mathcal{C}$ is abelian, $K(\mathcal{C})$ is no longer abelian in general [1, 5.7]. But $K(\mathcal{C})$ has **distinguished triangles**, which are triangles isomorphic to standard triangles.

11.3. **The cateogory $D(\mathcal{C})$.** We assume $\mathcal{C}$ is an abelian category. We then add some inverses of quasi isomorphisms in $K(\mathcal{C})$ to define $D(\mathcal{C})$. $D(\mathcal{C})$ again is not necessarily be an abelian category, but it is a **triangulated category** which has distinguished triangles which satisfy certain axioms.

By considering only complexes which are bounded below, we may define $C^+(\mathcal{C}), K^+(\mathcal{C}), D^+(\mathcal{C})$ etc.

PROPOSITION 11.4. [1, 4.8] *If $\mathcal{C}$ has enoufh injectives then $D^+(\mathcal{C})$ is equivalent to $K^+(I(\mathcal{C}))$, where $I(\mathcal{C})$ is the category of injective objects in $\mathcal{C}$.*

So, in a sence, to consider an object $X^\bullet$ of $D^+(\mathcal{C})$ is to consider an injective resolution $I^\bullet$ of $X^\bullet$ and treat it up to homotopy.

For left-exact functor $\mathcal{C}_1 \to \mathcal{C}_2$, we may "define" (the actual definiton should be done more carefully. See [1])

$$\mathbb{R}F : D^+(\mathcal{C}_1) \to D^+(\mathcal{C}_2)$$

by

$$\mathbb{R}F(X^\bullet) = F(I^\bullet)$$

where $I^\bullet$ is an injective resolution of $X^\bullet$.

A good thing about treating derived functors in this way is that we may easily treat derived functors of compositions:

$$\mathbb{R}(F \circ G) \cong (\mathbb{R}F) \circ (\mathbb{R}G).$$

REFERENCES

[1] P.P.Grivel, *Catégorie dérivées et foncteurs dérivés*, In: Algebraic D-modules, Perspectives in mathematics **2** (1997), 1–108.