# CATEGORIES, ABELIAN CATEGORIES AND COHOMOLOGIES.

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Resolutions and derived functors

Definition 6.1. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor between additive categories. We call $F$ **additive** if for any objects $M, N$ in $\mathcal{C}_1$,

$$\mathrm{Hom}(M, N) \to \mathrm{Hom}(F(M), F(N))$$

is additive.

Definition 6.2. Let $F$ be an additive functor from an abelian category $\mathcal{C}_1$ to $\mathcal{C}_2$.

(1) $F$ is said to be **left exact** (respectively, **right exact** ) if for any exact sequence

$$0 \to L \to M \to N \to 0,$$

the corresponding map

$$0 \to F(L) \to F(M) \to F(N)$$

(respectively,

$$F(L) \to F(M) \to F(N) \to 0)$$

is exact

(2) $F$ is said to be **exact** if it is both left exact and right exact.

Definition 6.3. Let $(K^\bullet, d_K)$, $(L^\bullet, d_L)$ be complexes of objects of an additive category $\mathcal{C}$.

(1) A **morphism of complex** $u : K^\bullet \to L^\bullet$ is a family

$$u^j : K^j \to L^j$$

of morphisms in $\mathcal{C}$ such that $u$ commutes with $d$. That means,

$$u^{j+1} \circ d_K^j = d_K^j \circ u^j$$

holds.

(2) A **homotopy** between two morphisms $u, v : K^\bullet \to L\bullet$ of complexes is a family of morphisms

$$h^j : K^j \to L^{j-1}$$

such that $u - v = d \circ h + h \circ d$ holds.

Lemma 6.4. *Let $\mathcal{C}$ be an abelian category that has enough injectives. Then:*

(1) *For any object $M$ in $\mathcal{C}$, there exists an* **injective resolution** *of $M$. That means, there exists an complex $I^\bullet$ and a morphism $\iota_M : M \to I^0$ such that*

$$H^j(I^\bullet) = \begin{cases} M \ (\text{via } \iota_M) & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}$$

(2) *For any morphism $f : M \to N$ of $\mathcal{C}$, and for any injective resolutions $(I^\bullet, \iota_M)$, $(J^\bullet, \iota_N)$ of $M$ and $N$ (respectively), There exists a morphism $\bar{f} : I^\bullet \to J^\bullet$ of complexes which commutes with $f$. Forthermore, if there are two such morphisms $\bar{f}$ and $f'$, then the two are homotopic.*

Definition 6.5. Let $\mathcal{C}_1$ be an abelian category which has enough injectives. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a left exact functor to an abelian category. Then for any object $M$ of $\mathcal{C}_1$ we take an injective resolution $I_M^\bullet$ of $M$ and define

$$R^i F(M) = H^i(I_M^\bullet).$$

and call it the derived functor of $F$.

Lemma 6.6. *The derived functor is indeed a functor.*