$\mathbb{Z}_p$, $\mathbb{Q}_p$, AND THE RING OF WITT VECTORS

No.04: $\boxed{\text{Review}}$

There are several ways to define $\mathbb{Z}_p$:

(1) $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to the p-adic metric d_p.

(2) $\mathbb{Z}_p$ is the projective limit $\varprojlim_k(\mathbb{Z}/p^k\mathbb{Z})$.

(3)
$$\mathbb{Z}_p = \{[0.a_0a_1a_2...]_p; a_i \in \{0, 1, 2, 3, \dots, p-1\}\}$$

Definition 4.1. We equip

$$\varprojlim_k(\mathbb{Z}/p^k\mathbb{Z}) = \{(b_j)_{j=1}^{\infty}; b_j \in \mathbb{Z}/p^j\mathbb{Z}, (b_{j_1} \text{ modulo } p^{j_2}) = b_{j_2} \text{ whenever } j_1 > j_2\}$$

with the following "p-addic norm".

$$|(b_j)|_p = \begin{cases} \frac{1}{p^k} & \text{if } b_k = 0 \text{ and } b_{k+1} \neq 0 \\ 0 & \text{if } (b_j) = 0 \end{cases}$$

Then we define "p-addic metric" d_p on $\varprojlim_k \mathbb{Z}/p^k\mathbb{Z}$ in an obvious way.

Lemma 4.2. *A natural map $\iota : (\mathbb{Z}, d_p) \to (\varprojlim_k(\mathbb{Z}/p^k\mathbb{Z}), d_p)$ defined by*

$$\iota : \mathbb{Z} \ni n \mapsto (n, n, n \dots) \in \varprojlim_k(\mathbb{Z}/p^k\mathbb{Z})$$

is an isometry.

Exercise 4.1. Prove the lemma above.