## $\mathbb{Z}_p$ , $\mathbb{Q}_p$ , AND THE RING OF WITT VECTORS

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No.02: definition of  $\mathbb{Z}_p$ 

Let p be a prime ("base"). We would like to introduce a metric on  $\mathbb{Z}$  such that

 $n : \text{small} \iff n \text{ is divisible by powers of } p.$ 

Namely:

Definition 2.1. Let p be a prime number.

(1) We define a *p*-adic norm  $| \bullet |_p$  on  $\mathbb{Z}$  as follows.

$$|n|_p = \begin{cases} \frac{1}{p^k} & \text{if } n \neq 0 \text{ and } p^k | n \text{ and } p^{k+1} \not| n \\ 0 & \text{if } n = 0 \end{cases}$$

(2) We define a p-adic distance  $d_p(\bullet, \bullet)$  on  $\mathbb{Z}$  as follows.

$$d_p(n,m) = |n - m|_p \qquad (n, m \in \mathbb{Z})$$

(1)  $| \bullet |_d$  enjoys the following properties. Lemma 2.2.

- (a)  $|x|_p \ge 0$  ( $\forall x \in \mathbb{Z}$ ).  $|x|_p = 0 \iff x = 0$ . (b)  $|x+y|_p \le \max(|x|_p, |y|_p)$  ( $\le (|x|_p + |y|_p)$ .
- (c)  $|xy|_p = |x|_p |y|_p$
- (2)  $(\mathbb{Z}, d_p)$  is a metric space.

DEFINITION 2.3. A metric space (X, d) is said to be **complete** if every Cauchy sequence of X converges to an element of X.

THEOREM 2.4. Let (X, d) be a metric space. There exists a complete metric space  $(\bar{X}, d)$  with an isometry  $\iota : X \to \bar{X}$  such that X is dense in X. Furthermore, X is unique up to a unique isometry.

DEFINITION 2.5. Let (X, d) be a metric space. We call (X, d) as in the above theorem the completion of (X, d).

DEFINITION 2.6. Let p be a prime number. We denote the completion of  $(\mathbb{Z}, d_p)$  by  $(\mathbb{Z}_p, d_p)$  and call it the ring of p-addic integers. Thus elements of  $\mathbb{Z}_p$  are p-addic integers.

Theorem 2.7.  $\mathbb{Z}_p$  has a unique structure of a topological ring. Namely,

(1) There exists unique continuous maps

$$+: \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$$

(addition) and

$$\times : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$$

(multiplication) which are extensions of the usual addition and multiplication of  $\mathbb{Z}$ .

(2)  $(\mathbb{Z}_p, +, \times)$  is a commutative associative ring.

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DEFINITION 2.8. Let p be a prime number. For any sequence  $\{a_j\}_{j=0}^{\infty}$  such that  $a_j \in \{0, 1, 2, 3, \dots, p-1\}$ , we consider a sequence  $\{s_n\}$  defined by

$$s_n = \sum_{j=0}^n a_j p^j.$$

Then the sequence  $\{s_n\}$  is a Cauchy sequence in  $\mathbb{Z}_p$ . We denote the limit of the sequence as

$$[0.a_0a_1a_2a_3\dots]_p.$$

EXERCISE 2.1. compute

$$[0.1]_3 + [0.2222\dots]_3$$