CONGRUENT ZETA FUNCTIONS. NO.5

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5.1. **Affine schemes.** We define affine schemes as a representable functor.

DEFINITION 5.1. Let R be a ring. Then we denote by $\mathrm{Spec}(R)$ the **affine scheme with coordinate ring** R.

For any affine scheme $\mathrm{Spec}(R)$ and for any ring S, we define the S**-valued point** of $\mathrm{Spec}(R)$ by

$$\mathrm{Spec}(R)(S) = \mathrm{Hom}_{\mathrm{ring}}(R, S)$$

LEMMA 5.2. *Let k be a ring. Let $\{f_1, f_2, \dots, f_m\}$ be a set of equations in n-variables $X_1, X_2, \dots, X_n$ over k. Let us put*

$$A = k[X_1, X_2, \dots, X_n]/(f_1, f_2, \dots, f_m).$$

Then we have a natural identification

$$V(f_1, f_2, \dots, f_m)(K) = \mathrm{Spec}(A)(K)$$

for any algebra K over k.

COROLLARY 5.3. *We employ the assumption as the Lemma. Then:*

(1) *When the "target algebra" K is given, the set of solutions $V(f_1, f_2, \dots, f_m)(K)$ depends only on the affine coordinate ring A.*
(2) *For any element $P \in \mathrm{Spec}(A)(K)$, the "evaluation map"*

$$A \ni f \mapsto \mathrm{eval}_P(f) \in K$$

is defined in an obvious way. Thus every element of A may be regarded as a K-valued function on $\mathrm{Spec}(A)(K)$.

5.2. **localization.**

DEFINITION 5.4. Let f be an element of a commutative ring A. Then we define the localization A_f of A with respect to f as a ring defined by

$$A_f = A[Y]/(Yf - 1)$$

where Y is a indeterminate.

LEMMA 5.5. *When K is a field, then we have a canonical identification*

$$\mathrm{Spec}(A_f)(K) = \{P \in \mathrm{Spec}(A)(K); \mathrm{eval}_P(f) \neq 0\}.$$