TOPICS IN NON COMMUTATIVE ALGEBRAIC GEOMETRY AND CONGRUENT ZETA FUNCTIONS (PART VI). LIE ALGEBRAS AND THEIR ENVELOPING ALGEBRAS

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1. Lie algebras

Definition 1.1. Let K be a commutative ring. Then a **Lie algebra $\mathfrak{g}$** over K is a K-module with a bilinear (non associative) bracket product ("Lie bracket")

$$[\bullet, \bullet] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which satisfies the following axioms:

(1) $[X, X] = 0$ for all $X \in \mathfrak{g}$.
(2) ("Jacobi identity")

$$[X, [Y, Z]] = [[X, Y], Z]] + [Y, [X, Z]] \qquad (\forall X, Y, Z \in \mathfrak{g}).$$

Example 1.2. Any associative algebra A over k may be regarded as a Lie algebra with the "commutator" as a Lie bracket.

In this talk, we always regard associative algebra as a Lie algebra equipped with the commutator product unless otherwise specified.

Lemma 1.3. *Let $\mathfrak{g}$ be a Lie algebra over a ring k. Then there exists an associative unital algebra $U(\mathfrak{g})$ with a Lie algebra homomorphism*

$$\iota_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$$

with the following universal property:

For any associative unital algebra A with a Lie algebra homomorphism $\phi : \mathfrak{g} \to A$, there exists a unique algebra homomorphism

$$\psi : U(\mathfrak{g}) \to A$$

such that $\psi \circ \iota_{\mathfrak{g}} = \phi$ holds.

The pair $(U(\mathfrak{g}), \iota_{\mathfrak{g}})$ is unique up to an isomorphism.

Definition 1.4. Under the assumption of the previous Lemma, The pair $(U(\mathfrak{g}), \iota_{\mathfrak{g}})$ is called the **universal enveloping algebra** the Lie algebra $\mathfrak{g}$.

Universal enveloping algebras of Lie algebras form an important class of non commutative associative algebras. Our task in this Part is to describe these algebras in our language.

2. Representations of a Lie algebra

Definition 2.1. Let k be a field. A finite dimensional representation of a Lie algebra $\mathfrak{g}$ over k is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to M_n(k).$$

Note: The full matrix algebra $M_n(k)$, when regarded as a Lie algebra equipped with the commutator product, is commonly denoted as $\mathfrak{gl}_n(k)$.

Example 2.2. Let k be a field. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over k. We then have an **adjoint representation**

$$\mathfrak{g} \ni X \mapsto \mathrm{ad}(X) = (Y \mapsto [X, Y]) \in \mathrm{End}_{k-\mathrm{linear}}(\mathfrak{g}).$$

3. Poincaré-Birkoff-Witt Theorem

In this section we prove the Poincaré-Birkoff-Witt Theorem. The treatment here essentially follows [1]. Let $\mathfrak{g}$ be a Lie algebra over a field k. To prove the theorem we consider $S_k(\mathfrak{g})$, the symmetric algebra of $\mathfrak{g}$ over k. Let us denote the multiplication of $S_k(\mathfrak{g})$ by $(x, y) \mapsto x \circ y$. We note that each element x of $S_k(\mathfrak{g})$ has its degree $\deg(x)$. (as a polynomial in elements of $\mathfrak{g}$.)

Lemma 3.1. *We choose a ordered basis $(x_\lambda; \lambda \in \Omega)$. (That means, a basis with a totally ordered index set Ω.) Then there exists a linear action of $\mathfrak{g}$ on $S_k(\mathfrak{g})$ which obeys the following rules:*

(1) *For any $x \in \mathfrak{g}$ and for any $y \in S_k(\mathfrak{g})$,*

$$\deg(x.y - x \circ y) \leq \deg(y)$$

(2) *If $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then we have*

$$x_{\lambda_0}.(x_{\lambda_1} \circ x_{\lambda_2} \circ x_{\lambda_3} \circ \cdots \circ x_{\lambda_n}) = x_{\lambda_0} \circ x_{\lambda_1} \circ x_{\lambda_2} \circ x_{\lambda_3} \circ \cdots \circ x_{\lambda_n}.$$

(3) *For any $x, y \in \mathfrak{g}$ and for any $z \in S_k(\mathfrak{g})$, we have*

$$x.(y.z) - y.(x.z) = [x, y].z$$

The proof is done by a careful use of induction. Namely,

Sublemma 3.2. *We employ the same assumption of the above Lemma. Then for each $m \in \mathbb{Z}_{>0}$, there exists a unique k-bilinear map*

$$f_m : \mathfrak{g} \times S_k(\mathfrak{g})_{\leq m} \to S_k(\mathfrak{g})_{\leq m+1}$$

which obeys the following rules:

(1) *For any* $x \in \mathfrak{g}$ *and for any* $y \in S_k(\mathfrak{g})_{\leq m}$,

$$\deg(f_m(x,y) - x \circ y) \leq \deg(y)$$

(2) *If* $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ *and* $n \leq m$*, then we have*

$$f_m(x_{\lambda_0}, x_{\lambda_1} \circ x_{\lambda_2} \circ x_{\lambda_3} \circ \cdots \circ x_{\lambda_n}) = x_{\lambda_0} \circ x_{\lambda_1} \circ x_{\lambda_2} \circ x_{\lambda_3} \circ \cdots \circ x_{\lambda_n}.$$

(3) *For any* $x, y \in \mathfrak{g}$ *and for any* $z \in S_k(\mathfrak{g})_{\leq m-1}$*, we have*

$$f_m(x, f_m(y,z)) = f_m(y, f_m(x,z)) + f_m([x,y],z)$$

Proof. We note first that $S_k(\mathfrak{g})$ has the set of monomials

$$\{x_{\lambda_1} \circ x_{\lambda_2} \circ x_{\lambda_3} \circ \cdots \circ x_{\lambda_n}; \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n\}$$

as a k-basis. For monomial $w = x_{\lambda_1} \circ x_{\lambda_2} \circ x_{\lambda_3} \circ \cdots \circ x_{\lambda_n}$ such that $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$, we put $z = \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$. Then

$$w = f_{m-1}(x_{\lambda_1}, z).$$

We define inductively the action of x_{λ_0} on it by the following equations.

$$f_m(x_{\lambda_0}, x_{\lambda_1} \circ z) = \begin{cases} x_{\lambda_0} \circ x_{\lambda_1} \circ z & (\text{if } \lambda_0 \leq \lambda_1) \\ \begin{pmatrix} x_{\lambda_1} \circ x_{\lambda_0} \circ z \\ + f_{m-1}(x_{\lambda_1}, f_{m-1}(x_{\lambda_0}, z) - x_{\lambda_0} \circ z) \\ + f_{m-1}([x_{\lambda_0}, x_{\lambda_1}], z) \end{pmatrix} & (\text{if } \lambda_0 > \lambda_1) \end{cases}$$

We first note that the above definition is necessary to meet our conditions. Indeed, by (2) we necessarily define as above for $\lambda_0 \leq \lambda_1$. When $\lambda_0 > \lambda_1$, we compute

$$\begin{aligned} & x_{\lambda_0}.(x_{\lambda_1} \circ z) \\ \overset{(3)}{=} & x_{\lambda_1}.x_{\lambda_0}.z + [x_{\lambda_0}, x_{\lambda_1}].z \\ = & x_{\lambda_1}.(x_{\lambda_0}.z - x_{\lambda_0} \circ z) + x_{\lambda_1}.(x_{\lambda_0} \circ z) + [x_{\lambda_0}, x_{\lambda_1}].z \\ \overset{(2)}{=} & x_{\lambda_1}.(x_{\lambda_0}.z - x_{\lambda_0} \circ z) + x_{\lambda_1} \circ x_{\lambda_0} \circ z + [x_{\lambda_0}, x_{\lambda_1}].z \end{aligned}$$

and take a careful look at degrees of each monomials using (1). From this argument we see in particular that the action is uniquely determined by conditions (1),(2),(3).

It is easy to see that the conditions (1),(2) are satisfied by f_m defined as above.. Let us proceed to verify that the f_m so defined also satisfies (3). Let us consider x_λ, x_μ $z = x_{\mu_1} \circ x_{\mu_2} \circ \cdots \circ x_{\mu_n}$ with $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, $n \leq m-1$. We need to prove

$$x_\lambda.x_\mu.z - x_\mu.x_\lambda.z = [x_\lambda, x_\mu].z. \tag{$\flat$}$$

Since the equation above is antisymmetric in μ, ν, we may assume that $\lambda \leq \mu$.

(i) Case where $\lambda \leq \mu_1$.

$$\begin{aligned} & x_\lambda . x_\mu . z \\ =& x_\lambda . (x_\mu \circ z) + x_\lambda . (x_\mu . z - x_\mu \circ z) \\ \overset{(1)}{=} & x_\lambda \circ x_\mu \circ z + x_\lambda . (x_\mu . z - x_\mu \circ z) \end{aligned}$$

In other words,

$$f_m(x_\lambda, f_m(x_\mu, z)) = x_\lambda \circ x_\mu \circ z + f_{m-1}(x_\lambda, (f_{m-1}(x_\lambda, z) - x_\mu \circ z)).$$

On the other hand we have

$$\begin{aligned} & x_\mu . x_\lambda . z \\ =& x_\mu . (x_\lambda \circ z) \\ \overset{\text{by def}}{=} & x_\lambda \circ x_\mu \circ z + f_{m-1}(x_\lambda, f_{m-1}(x_\mu, z) - x_\mu \circ z) + f_{m-1}([x_\mu, x_\lambda], z) \end{aligned}$$

So the equation $\flat$ surely holds in this case.
(ii) Case where $\lambda, \mu > \mu_1$.

In this case we need to "decompose" z further:

$$z = x_\nu . w.$$

We first forget about the hypothesis $\lambda \leq \mu$ and prove

$$\begin{aligned} & x_\lambda . (x_\mu . (x_\nu . w)) \qquad (\heartsuit) \\ =& x_\nu . (x_\lambda . (x_\mu . w)) + [x_\lambda, x_\nu] . (x_\mu . w) + [x_\mu, x_\nu] . (x_\lambda . w) + [x_\lambda, [x_\mu, x_\nu]] . w \end{aligned}$$

(Since we are doing induction, we need to pay a special attention on degrees on operands. That means, we should use f_m's rather than the above "lazy" notation. But that is fairly cumbersome, so we keep on being lazy here.)

Let us now admit that the above equation $\heartsuit$ is true and prove the rest of the equation (3). By interchanging λ and μ in the equation ($\heartsuit$), we obtain

$$\begin{aligned} & x_\mu . (x_\lambda . (x_\nu . w)) \qquad (\diamondsuit) \\ =& x_\nu . (x_\mu . (x_\lambda . w)) + [x_\mu, x_\nu] . (x_\lambda . w) + [x_\lambda, x_\nu] . (x_\mu . w) + [x_\mu, [x_\lambda, x_\nu]] . w \end{aligned}$$

Then by subtracting ($\diamondsuit$) from ($\heartsuit$), we obtain

$$\begin{aligned} & x_\lambda . (x_\mu . (x_\nu . w)) - x_\mu . (x_\lambda . (x_\nu . w)) \\ =& x_\nu . (x_\lambda . (x_\mu . w) - x_\mu . (x_\lambda . w)) \\ & + ([x_\lambda, [x_\mu, x_\nu]] - [x_\mu, [x_\lambda, x_\nu]]) . w. \end{aligned}$$

Since $\deg(w)$ is smaller than $\deg(z)$, by induction hypothesis the first term in the right hand side may be replaced by $x_\nu.([x_\lambda, x_\mu].w)$. The second term may be replaced, by the Jacobian identity, by $[[x_\lambda, x_\mu], x_\nu]$. So the equation ($\flat$) holds in this case too.

It remains to prove the equation ($\heartsuit$). By the induction hypothesis we have

$$x_\mu.(x_\nu.w) = x_\nu.(x_\mu.w) + [x_\mu, x_\nu].w.$$

Also by the induction hypothesis we have

$$x_\lambda.([x_\mu, x_\nu].w) = [x_\mu, x_\nu].(x_\lambda.w) + [x_\lambda, [x_\mu, x_\nu]].w$$

Lastly, we decompose $x_\mu.w$ as

$$x_\mu.w = (x_\mu \circ w) + (x_\mu.w - x_\mu \circ w). = (x_\mu \circ w) + y$$

Then the second term y has degree smaller than $\deg(z) = \deg(w) + 1$. The case (i) applies to the first term and we obtain:

$$x_\lambda.(x_\nu.(x_\mu.w)) = x_\nu.(x_\lambda.(x_\mu.w)) + [x_\lambda, x_\nu].(x_\mu.w).$$

These altogether complete the proof. $\square$

THEOREM 3.3 (Poincaré, Birkoff, Witt(PBW)). *Let $\mathfrak{g}$ be a Lie algebra over a field k. Then we have a k-algebra isomorphism*

$$\Psi : \mathrm{Gr}(U(\mathfrak{g})) \cong S(\mathfrak{g}).$$

PROOF. Let

$$\iota_0 : \mathfrak{g} \to \mathrm{Gr}(U(\mathfrak{g}))$$

be the obvious k-linear map.

Using the universality of symmetric algebra, there exists a unique k-algebra homomorphism

$$\Phi : S(\mathfrak{g}) \to \mathrm{Gr}(U(\mathfrak{g}))$$

which extends ι_0. On the other hand the action defined in the Lemma 1.3 gives us a linear map

$$\Psi_0 : U(\mathfrak{g}) \ni x \mapsto x.1 \in S(\mathfrak{g})$$

which is clearly degree-decreasing. So it defines a k-linear map

$$\Psi : \mathrm{Gr}(U(\mathfrak{g})) \to \mathrm{Gr}(S(\mathfrak{g})) \cong S(\mathfrak{g}).$$

Now the composition we obtain

$$\Psi \circ \Phi : S(\mathfrak{g}) \xrightarrow{\Phi} \mathrm{Gr}(U(\mathfrak{g})) \xrightarrow{\Psi} S(\mathfrak{g})$$

coincides with the identity map. Indeed, it coincides with the identity on monomials of the form

$$x_{\lambda_1} \circ x_{\lambda_2} \circ x_{\lambda_3} \circ \cdots \circ x_{\lambda_{n-1}} \circ x_{\lambda_n}.$$

The map Φ is easily verified to be surjective. So we conclude that Φ and Ψ are both bijective and are inverse to each other. $\square$

4. Jordan-Chevalley decomposition of a square matrix

4.1. **Existence and uniqueness of Jordan-Chevalley decomposition.**

Definition 4.1. Let A be a square matrix over a field k. A **Jordan-Chevalley decomposition** (also called SN-decomposition) of A is a decomposition of A

$$A = S + N$$

which satisfies the following conditions.

(1) S is semisimple (that means, the minimal polynomial of S has only simple roots.)
(2) N is nilpotent.
(3) $SN = NS$
(4) $S, N \in \overline{k}[A]$

A main objective of this section is to prove the following proposition.

Proposition 4.2. *For any square matrix A over a field k, there exists a unique Jordan-Chevalley decomposition.*

To prove it, we need some basic facts from linear algebra.

Lemma 4.3. *Let A be a square matrix over a field k. Let $m_A(X)$ be the minimal polynomial of A over k. If m_A is decomposed into two coprime polynomials, that means,*

$$m_A = m_1 m_2 \qquad (m_1, m_2) = 1,$$

then A is similar to a direct sum

$$A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where $m_1(A_1) = 0$, $m_2(A_2) = 0$.

Proof. Since $k[X]$ is an Euclidean domain, it is a principal ideal domain. thus we see that there exists a polynomial $l_1(X), l_2(X) \in k[X]$ such that

$$l_1(X)m_1(X) + l_2(X)m_2(X) = 1$$

holds. We put

$$E_j = l_j(A)m_j(A) \qquad (j = 1, 2).$$

Then we see easily that E_1, E_2 are mutually orthogonal projection. That means, we have

$$E_1^2 = E_1, E_2^2 = E_2, E_1 + E_2 = 1, E_1E_2 = 0.$$

It is also easy to see that both E_1 and E_2 commute with A. Now putting $A_1 = A|_{\text{Range } E_2}$ and $A_2 = A|_{\text{Range } E_1}$ we see that

$$A = AE_2 + AE_1 = A_1 \oplus A_2.$$

with A_1 and A_2 satisfying the required property. □

Corollary 4.4. *Every square matrix A over a field k is similar to a direct sum of square matrices $A_1, A_2, \ldots, A_s$ with each minimal polynomial m_{A_j} equals to a power $f_j^{e_j}$ of a irreducible polynomial f_j over k.*

□

Corollary 4.5. *When k is algebraically closed, every square matrix A over a field k is similar to a direct sum of square matrices $A_1, A_2, \ldots, A_s$ with each minimal polynomial m_{A_j} equals to a power $(X - c_j)^{e_j}$ of a polynomial of degree 1 over k.*

□

Corollary 4.6. *A square matrix over a field k is semisimple if and only if it is diagonalizable (similar to a diagonal matrix) over $\overline{k}$.*

□

Corollary 4.7. *Let S_1, S_2 be semisimple square matrices of the same size over k. if S_1 and S_2 commute, then both $S_1 + S_2$ and $S_1 S_2$ are also semisimple.*

Proof. Using commutativity of S_1 and S_2, we may easily see that S_1 and S_2 are simultaneously diagonalizable over $\overline{k}$. □

Corollary 4.8. *Let k be a field. Let C be a commutative subalgebra of $M_n(k)$. If C is generated by semisimple elements, then every element of C is also semisimple.*

□

On the other hand we have

Lemma 4.9. *Let k be a field. Let C be a commutative subalgebra of $M_n(k)$. If C is generated by nilpotent elements, then every element of C is also nilpotent.*

Proof. Easy. □

Corollary 4.10. *A Jordan-Chevalley decomposition (if there exists) of a square matrix A is unique.*

Proof. Let

$$A = S + N = S' + N'$$

be two Jordan-Chevalley decompositions. Then $S - S' = N' - N$ is a semisimple nilpotent element. Thus $S - S' = N' - N = 0$. □

Proof. (of Proposition 4.2.) It now remains to prove that Jordan-Chevalley decomposition of a square matrix exists. By definition we may assume that k is algebraically closed. In view of Corollary 4.5, we may then assume that the minimal polynomial m_A of A is of the form $(X - c)^e$ for some $c \in k$ and $e \in \mathbb{Z}_{>0}$. Then

$$A = c + (A - c)$$

gives the required Jordan-Chevalley decomposition.

□

Definition 4.11. Let k be a field. For any square matrix $x \in M_n(k)$, we denote by x_s (respectively, x_n) the semisimple (respectively, nilpotent) part of x in the Jordan-Chevalley decomposition of x.

Lemma 4.12. *Let k be a field. Let $x \in M_n(k)$ be a square matrix. then we have*

$$(\mathrm{ad}(x))_s = \mathrm{ad}(x_s), \quad (\mathrm{ad}(x))_n = \mathrm{ad}(x_n)$$

Proof. Follows easily from the uniqueness of the Jordan-Chevalley decomposition. □

4.2. k-rationality.

Proposition 4.13. *Let A be a square matrix over a field k. Let*

$$A = S + N$$

be the Jordan-Chevalley decomposition of A.

Let m_A be the minimal polynomial of A. If all of the roots of m_A are separable over k, then S and N are defined over k. (That means, they are matrices over k).

Proof. In view of Corollary , we may assume that m_A is of the form f^e for some irreducible polynomial f and a positive integer e. By assumption, f has only simple roots.

$$f(X) = \prod_{j=1}^{d} (X - c_j)$$

Let us define a polynomial $\chi_s(X) \in \overline{k}[X]$ as follows.

$$\chi_s(X) = \frac{\prod_{\substack{1 \le j \le d \\ j \neq s}} (X - c_j)}{\prod_{\substack{1 \le j \le d \\ j \neq s}} (c_s - c_j)} \qquad (s = 1, 2, 3, \dots, d)$$

These polynomials are designed to satisfy the following property.

$$\chi_s(c_j) = \begin{cases} 1 & \text{if } j = s \\ 0 & \text{if } j \neq s \end{cases}$$

Then we further define

$$\phi_s^{(e)}(X) = 1 - (1 - \chi_s^e)^e$$

and

$$\psi(X) = \sum_{s=1}^{d} c_s \phi_s^{(e)}(X).$$

It is fairly easy to see that

$$S = \psi(A)$$

holds.

The function ψ is symmetric with respect to roots $\{c_s\}$ and thus ψ is a polynomial with coefficients in k. Thus S (hence also N) is defined over k.

$\square$

The following example shows that the k-rationality of S does not necessarily hold when we drop off the assumption on A.

Example 4.14. Let $A \in M_p(\mathbb{F}_p(x))$ be a matrix of the following form.

$$A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ x & & & 0 \end{pmatrix}$$

Then the minimal polynomial of A is given by $X^p - x$. The Jordan-Chevalley decomposition of A is given by

$$A = x^{1/p} + (A - x^{1/p}).$$

Thus the decomposition is not defined over $\mathbb{F}_p(x)$.

5. GENERALITIES IN FINITE DIMENSIONAL LIE ALGEBRAS

5.1. Ideals of Lie algebras.

DEFINITION 5.1. For a linear subspace S, T of a Lie algebra L, let us define denote by $[S, T]$ the following linear subspace of L.

(1)
$$[S, T] = (\text{linear span of } \{[x, y]; x \in S, y \in T\})$$

DEFINITION 5.2. Let L be a Lie algebra over a field k. A k-linear subspace $\mathfrak{a}$ of L is said to be an **ideal** of L if

$$[x, y] \in \mathfrak{a} \qquad (\forall x \in L, \forall y \in \mathfrak{a})$$

holds. This clearly is equivalent to saying that

$$[L, \mathfrak{a}] \subset L$$

holds.

PROPOSITION 5.3. *Let $\mathfrak{a}$ be an ideal of a Lie algebra L. Then*

(1) *$\mathfrak{a}$ is a sub L-module (sub representation) of L.*
(2) *$L/\mathfrak{a}$ caries a natural structure of a Lie algebra.*

PROOF. As usual. □

5.2. Simple, semisimple, solvable, and nilpotent Lie algebras: definition.

DEFINITION 5.4. For a Lie algebra L, let us define the following ideals of L.

(1) $\mathrm{Comm}(L) = [L, L]$, and inductively,
$$\mathrm{Comm}^j(L) = \mathrm{Comm}(\mathrm{Comm}^{j-1}(L)).$$
(2) $\mathrm{ad}(L)(L) = [L, L]$, and inductively,
$$\mathrm{ad}^j(L)(L) = \mathrm{ad}(\mathrm{ad}^{j-1}(L)).$$

LEMMA 5.5. *For any Lie algebra L and for any positive integer j, we have*
$$\mathrm{Comm}^j(L) \subset \mathrm{ad}^j(L).$$

PROOF. Inductively, we have
$$\mathrm{Comm}^j(L) = [\mathrm{Comm}^{j-1}(L), \mathrm{Comm}^{j-1}(L)] \subset [L, \mathrm{ad}^{j-1}(L)] = \mathrm{ad}^j(L).$$
□

DEFINITION 5.6. A Lie algebra L over a field k is said to be

(1) **semisimple** if it has no abelian ideals.
(2) **simple** if it has no non trivial ideals and $\dim(L) > 1$.

(3) **solvable** if $\mathrm{Comm}^N(L) = 0$ for some $N \in \mathbb{Z}_{>0}$.
(4) **nilpotent** if $\mathrm{ad}(L)^N(L) = 0$ for some $N \in \mathbb{Z}_{>0}$.

PROPOSITION 5.7. *We have the following implications.*
(1) *Simple Lie algebras are semisimple.*
(2) *Nilpotent Lie algebras are solvable.*

PROOF. (1)is Easy. (2) follows from Lemma 5.5. □

Semisimple algebras and solvable ones are "orthogonal". For now we only mention the following

LEMMA 5.8. *Non zero solvable algebra L cannot be semisimple.*

PROOF. Let N_0 be a positive integer such that

$$\mathrm{Comm}_0^N(L) \neq 0, \qquad \mathrm{Comm}^{N_0+1}(L) = 0.$$

Then $\mathrm{Comm}^{N_0}(L)$ is a non-zero abelian ideal of L. □

5.3. The radicals of Lie algebras.

DEFINITION 5.9. A **radical** of a Lie algebra L is a maximal solvable ideal of L.

LEMMA 5.10. *Let $\mathfrak{a}$ be an ideal of a Lie algebra L. If $L/\mathfrak{a}$ and $\mathfrak{a}$ are both solvable Lie algebras, then L is also solvable.*

PROOF. Since $L/\mathfrak{a}$ is solvable, there exists a positive integer N_1 such that

$$\mathrm{Comm}^{N_1}(L/\mathfrak{a}) = 0.$$

Then we obviously have

$$\mathrm{Comm}^{N_1}(L) \subset \mathfrak{a}.$$

On the other hand, since $\mathfrak{a}$ is solvable, there exists a positive integer N_2 such that

$$\mathrm{Comm}^{N_2}(\mathfrak{a}) = 0.$$

We thus have

$$\mathrm{Comm}^{N_1+N_2}(L) = \mathrm{Comm}^{N_2}(\mathrm{Comm}^{N_1}(L)) \subset \mathrm{Comm}^{N_2}(\mathfrak{a}) = 0.$$

□

LEMMA 5.11. *Every Lie subalgebras and quotients of solvable Lie algebras are solvable.*

PROOF. Obvious. □

LEMMA 5.12. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of a Lie algebra L. If $\mathfrak{a}, \mathfrak{b}$ are both solvable (as Lie algebras), then $\mathfrak{a} + \mathfrak{b}$ is also solvable.*

Proof.

$$\mathfrak{a}+\mathfrak{b}/\mathfrak{b} \cong \mathfrak{a}/\mathfrak{a}\cap\mathfrak{b}$$

□

Proposition 5.13. *For a finite dimensional Lie algebra L over a field k, there exists a unique maximal solvable ideal of L. So we may call it* **the** *radical of L.*

Proof. Let $\mathfrak{a}_0$ be a solvable ideal of L which has the maximal dimension among solvable ideals. Then for any solvable ideal $\mathfrak{b}$ of L, $\mathfrak{a}_0+\mathfrak{b}$ is also solvable. Thus by the choice of $\mathfrak{a}_0$ we see that

$$\mathfrak{a}_0+\mathfrak{b}=\mathfrak{a}_0. \quad (\text{That is, } \mathfrak{a}_0 \supset \mathfrak{b}.)$$

Thus we see that $\mathfrak{a}_0$ is the largest solvable ideal of L.

□

Corollary 5.14. *Let L be a finite dimensional Lie algebra over a field k. Let $\mathfrak{r}$ be its radical. Then:*

(1) *$L/\mathfrak{r}$ is semisimple.*
(2) *L is semisimple if and only if $\mathfrak{r}=0$.*
(3) *A quotient $L/\mathfrak{a}$ is semisimple if and only if $\mathfrak{r}\subset\mathfrak{a}$.*

Proof. (1) follows immediately from the definition and Lemma 5.8.
(2) is also easy.
(3): $L/\mathfrak{a}$ contains

$$(\mathfrak{r}+\mathfrak{a})/\mathfrak{a}(\cong \mathfrak{r}/(\mathfrak{r}\cap\mathfrak{a}))$$

as a solvable ideal. □

5.4. Theorem of Engel.

Lemma 5.15. *Let k be a commutative ring. Let $x \in M_n(k)$ be a nilpotent matrix. Then $\operatorname{ad}(x)$ is also nilpotent.*

Proof. Assume $x^N=0$. We decompose $\operatorname{ad}(x)$ into left and right multiplication. Namely,

$$\operatorname{ad}(x)=\lambda(x)-\rho(x)$$

Then $\lambda(x)$ and $\rho(x)$ commute with each other.

$$\operatorname{ad}(x)^{2N-1}=\sum_{j=0}^{2N-1}\lambda(x)^j(-\rho(x))^{2N-1-j}=\sum_{j=0}^{2N-1}\lambda(x^j)\rho((-x)^{2N-1-j})=0.$$

□

Theorem 5.16 (Engel). *Let V be a finite dimensional vector space over a field k. Let L be a Lie subalgebra of $\mathfrak{gl}(V)$ such that each member of L is a nilpotent matrix. Then:*

(1) *If* $\dim(L) \geq 1$*, then* L *has an ideal of codimension* 1*.*
(2) *If* $\dim(V) \geq 1$*, then* L *has a simultaneous* 0*-eigen vector* v*. (That is,* $x.v = 0 \qquad (\forall x \in L), \quad v \neq 0.$*)*

PROOF. If $\dim(L) = 0$, then there is nothing to do. We proceed by induction on $\dim(L)$. Let L_1 be a maximal among

$$\{(\text{Lie subalgebras of } L \text{ which is not equal to } L)\}.$$

(The set above has a $\{0\}$ as a member, so it is not empty.) In view of the lemma above, we see

$$\forall x \in L_1 \exists N \in \mathbb{Z}_{>0} (\mathrm{ad}(x)^N(L) = 0).$$

We note that an vector space L/L_1 admits adjoint actions by L_1. Thus

$$\forall x \in L_1 \exists N \in \mathbb{Z}_{>0} (\mathrm{ad}(x)^N(L/L_1) = 0).$$

That means, $\mathrm{ad}_{L/L_1}(L_1) \subset \mathfrak{gl}(L/L_1)$ also satisfy the assumption of the theorem. By the induction hypothesis, we see that conclusion (2) is applicable to this case. Namely, there exists an element $y_0 \in L \setminus L_1$ such that

$$\mathrm{ad}(x)(y_0)(= [x, y_0]) \in L_1 \qquad (\forall x \in L_1)$$

holds. Now a vector subset

$$L_2 = k.y_0 + L_1 (\supsetneq L_1)$$

of L is closed under Lie bracket and therefore it is a Lie subalgebra of L. By the maximality of L_1, L_2 should equal to L. It is then also easy to verify that L_1 is an ideal of $L(= L_2)$. This proves (1).

To prove (2), we note that (L_1, V) satisfies the assumption of the theorem. So again by induction we see that L_1 has a simultaneous 0-eigen vector. In other words,

$$V_1 := \bigcap_{x \in L_1} \{v \in V; x.v = 0\} \supsetneq \{0\}.$$

Let us then consider the action of y_0.

$$v \in V_1 \implies x.(y_0.v) = y_0.(x.v) + [x, y_0].v = 0 \implies y_0.v \in V_1$$

Thus V_1 admits an action of y_0. Since y_0 is nilpotent on V by the assumption, we see that y_0 has at least one 0-eigen vector $v_0 (\neq 0)$ in V_1. Then v_0 surely is a simultaneous 0-eigen vector of L.

□

THEOREM 5.17 (Engel). *Let* V *be a finite dimensional vector space over a field* k*. Let* L *be a Lie subalgebra of* $\mathfrak{gl}(V)$ *such that each member* *of L is a nilpotent matrix. Then there exists a basis $e_1, e_2, \dots, e_n$ of V such that*

$$L \subset \mathfrak{n}_n(k) = \left\{ \begin{pmatrix} 0 & * & * & \dots & * \\ 0 & 0 & * & \dots & * \\ & & \ddots & * & * \\ 0 & 0 & \dots & 0 & * \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \right\}$$

holds with respect to this basis. In particular, L is nilpotent.

Proof. We apply the above theorem inductively to a vector space

$$L, L/(k.e_1), L/(k.e_1 + k.e_2) \dots$$

and obtain the desired basis $\{e_i\}$. Since the Lie algebra $\mathfrak{n}_n(k)$ is nilpotent, L is also nilpotent.

□

5.5. **Ideals of $\mathfrak{gl}_n(k)$.** Recall that $\mathfrak{n}_n(k)$ denotes the Lie algebra of strictly upper triangular matrices. In this subsection we denote by e_{ij} the elementary matrices. (as we have done so without even mentioning...)

Lemma 5.18. *Let k be a field of characteristic p (possibly 0). Let $n \in \mathbb{Z}_{>1}$. We assume that $(p, n) \neq (2, 2)$. Then we have*

$$\{x \in \mathfrak{gl}_n(k); [x, y] \in k.1_n \quad (\forall y \in \mathfrak{n}_n(k))\} = k.1_n + k.e_{1n}.$$

Proof. Let us denote by L the left hand side of the lemma. Then we trivially have $L \supset k.1_n$. Furthermore, for all $x \in \mathfrak{n}_n(k)$, we see easily that

$$xe_{1n} = 0, \qquad e_{1n}x = 0$$

holds. So we have

$$L \supset k.1_n + k.e_{1n}$$

Let us prove the opposite inclusion. We take an arbitrary element $x \in L$.

For any (i, j) satisfying $i < j$, we have $e_{ij} \in \mathfrak{n}_n(k)$ and thus

$$[e_{i,j}, x] = c1_n \qquad (\exists c \in k).$$

The rank of the left hand side is at most 2. So c must be equal to 0 when $n \geq 3$. Otherwise ($n = 2$), we compare the trace of the both hand sides. The trace of the left hand side is clearly zero. The trace of a scalar matrix $c.1_2$ is equal to $2c$. Thus $c = 0$ by our assumption ($(p, n) \neq (2, 2)$). In either case, we have $[e_{i,j}, x] = 0$. Then we compute some of special cases. First, let us examine the case where $i = 1, j \geq 2$. Then

$$0 = [e_{1j}, x] = \sum_{st} [e_{1j}, x_{st} e_{st}] = \sum_t x_{jt} e_{1t} - \sum_s x_{s1} e_{sj}$$

By looking at $(1, u)$ entry of the above equation, we conclude that equations in entries

$$\forall j \forall u ((j \geq 2, j \neq u) \implies x_{ju} = 0)$$
$$\forall j ((j \geq 2) \implies x_{jj} = x_{11})$$

hold. Similarly, by looking at the (u, n) entry of $[e_{in}, x]$, we conclude that equations

$$\forall i \forall u ((i \leq n-1, i \neq u) \implies x_{ui} = 0)$$

hold. Putting the equations all together, we conclude that x is in the right hand side of the lemma. $\square$

As an application of the Engel's theorem, we prove the following proposition.

PROPOSITION 5.19. *Let k be a field of characteristic p (possibly 0). Let n be a positive integer. We assume that $(p, n) \neq (2, 2)$. Then each ideal I of $\mathfrak{gl}_n(k)$ is equal to the one in the following list.*

(1) 0.
(2) $k.1_n$.
(3) $\mathfrak{sl}_n(k)$.
(4) $\mathfrak{gl}_n(k)$.

PROOF. The case $n = 1$ is trivial. So let us assume $n \geq 2$.

If $I \subset k.1_n$, then $\dim_k(I) \leq 1$ and hence $I = 0$ or $I = k.1_n$. Assume now $I \not\subset k.1_n$. Let us consider the Lie algebra $\mathfrak{n}_n(k)$ of strictly upper triangular matrices. Then

$$(L, V) = (\mathfrak{n}_n(k),\ (I + k.1_n)/k.1_n)$$

satisfies the assumption of the Engel's theorem. So there exists a non-constant element $x \in I$ such that

$$[x, y] \in k.1_n \qquad (\forall y \in \mathfrak{n}_n(k)).$$

holds. By using the previous lemma, we see that x may be presented as

$$x = c_0 1_n + c_1 e_{1n} \qquad (\exists c_0, c_1 \in k).$$

Since x is non-constant, we have $c_1 \neq 0$.

$$I \ni [e_{11}, x] = c_1 e_{1n}$$

Thus e_{1n} belongs to I. By changing the order of the base and repeating the above argument, we conclude that

$$\forall i \forall j ((i \neq j) \implies e_{ij} \in I.)$$

In addition we have

$$I \ni [e_{ij}, e_{ji}] = e_{ii} - e_{jj}$$

This clearly proves $I \supset \mathfrak{sl}_n(k)$. Since the codimension of $\mathfrak{sl}_n(k)$ in $\mathfrak{gl}_n(k)$ is 1, we have either $I = \mathfrak{sl}_n(k)$ or $I = \mathfrak{gl}_n(k)$.

□

For the sake of completeness, we deal with the case $(p, n) = (2, 2)$. In this case, situation is a bit different.

LEMMA 5.20. *Let k be a field of characteristic 2. Then:*

(1) *Any two-dimensional vector subspace V of $\mathfrak{sl}_2(k)$ with $V \supset k.1_2$ is equal to a vector space $L_{[b:c]}$ given by an element $[b : c] \in \mathbb{P}^1(k)$ which is defined as*

$$L_{[b:c]} = k.1_n + k. \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

(2) *For any element $[b : c] \in \mathbb{P}^1(k)$, the vector space $L_{[b:c]}$ is an ideal of $\mathfrak{gl}_2(k)$.*

(3) *In particular, $\mathfrak{t}_2(k) = k.1_2 + \mathfrak{n}_2(k)$ is an ideal of $\mathfrak{gl}_2(k)$.*

PROOF. (1) There exists a traceless non constant matrix $x \in V$ such that

$$I = k.1_n + k.x$$

holds. By subtracting a constant matrix, one may easily replace x by a matrix with zero diagonals.

(2) By a direct computation we see

$$\left[\begin{pmatrix} x & y \\ z & w \end{pmatrix}, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right] = (x-w) \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + (bz-cy)1_2 \qquad (x, y, z, w, b, c, \in k)$$

(Note that $\operatorname{char}(k) = 0$.)

(3) $\mathfrak{t}_2(k) = L_{[1:0]}$. □

PROPOSITION 5.21. *Let k be a field of characteristic 2. Then each ideal I of $\mathfrak{gl}_2(k)$ is equal to the one in the following list.*

(1) 0.
(2) $k.1_n$.
(3) *A two-dimensional Lie algebra $L_{[b:c]}$ defined as in the lemma above.*
(4) $\mathfrak{sl}_n(k)$.

(5) $\mathfrak{gl}_n(k)$.

PROOF. We divide into several cases.
(i) Case where $I \not\subset \mathfrak{sl}_2(k)$. In this case there exists an element $x \in I$ with $\operatorname{tr}(x) \neq 0$. Putting

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we compute as follows

$$I \ni [e_{12}, x] = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \begin{pmatrix} c & a+d \\ 0 & c \end{pmatrix}.$$

(Note $\operatorname{char}(k) = 2$.) Then we have

$$I \ni [e_{11}, [e_{12}, x]] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & a+d \\ 0 & c \end{pmatrix} \right] = \begin{pmatrix} 0 & a+d \\ 0 & 0 \end{pmatrix}$$

Thus we see that $e_{12} \in I$. In a same way (by changing the order of the base), we obtain, $e_{21} \in I$.

$$1_2 = [e_{21}, e_{12}] \in I.$$

Since $\operatorname{tr}(x) \neq 0$, we see that $\{1_2, e_{12}, e_{21}, x\}$ spans the $\mathfrak{gl}_2(k)$. thus $I = \mathfrak{gl}_2(k)$ in this case.
(ii) Case where $I \subset \mathfrak{sl}_2(k)$ and $I \cap k.1_2 = 0$. Let x be arbitrary element of I and put

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then by computing $[e_{12}, x]$ as in the case (i) above, we know that $c = 0$. Similarly, we know $b = 0$. Since x is traceless, $a = d$ also holds. So the only possibility in this case is $I = 0$.
(iii) Case where $k.1_2 \subsetneq I \subsetneq \mathfrak{sl}_2(k)$. By a dimension consideration, we see that $\dim I = 2$. Then we use the above lemma.
(iv) The case $I = k.1_2$ or $I = \mathfrak{sl}_2(k)$. Excellent. There is nothing to in this case.

□

5.6. **Ideals of $\mathfrak{sl}_n(k)$.**

PROPOSITION 5.22. *Let k be a field of characteristic p (possibly 0.) Let n be a positive integer.*

(1) *If $p \not| n$, then $\mathfrak{sl}_n(k)$ is a simple Lie algebra.*
(2) *If $p | n$, then $\mathfrak{sl}_n(k)$ has a unique nontrivial ideal $k.1_n$.*

PROOF. Let I be an ideal of $\mathfrak{sl}_n(k)$. By taking trace we see immediately that

$$1_n \in \mathfrak{sl}_n(k) \iff p|n.$$

Thus if $I \subset k.1_n$, then The only nontrivial possibility is that $p|n$ and $I = k.1_n$.

Assume now that $I \not\subset k.1_n$. Then by an argument similar to that in Proposition 5.19, we see that

$$x = c_0 1_n + c_1 e_{1n} \in I, \quad (\exists c_0, c_1 \in k, c_1 \neq 0)$$

holds.

(1) If $p \not| n$, then by taking trace we see that $e_{1n} \in I$. By permuting the basis, we see that $e_{ij} \in I$ whenever $i \neq j$. Thus $I = \mathfrak{sl}_n(k)$ in this case.

(2) If $p|n$, then by assumption on (p, n) we have $n \geq 3$. Thus

$$I \ni [e_{21}, x] = c_1 e_{2n} \quad \therefore e_{1n} = [e_{12}, e_{2n}] \in I.$$

So in this case also we see that $I = \mathfrak{sl}_n(k)$.

□

PROPOSITION 5.23. *Let k be a field of characteristic 2. Then any two dimensional Lie algebra $L_{[b:c]}$ as in Lemma 5.20 is an ideal of $\mathfrak{sl}_2(k)$. Thus each ideal I of $\mathfrak{sl}_2(k)$ is equal to the one in the following list.*

(1) 0.
(2) $k.1_n$.
(3) $L_{[b:c]}$
(4) $\mathfrak{sl}_n(k)$

PROOF. Easy exercise. □

5.7. **Invariant bilinear forms and Killing forms.**

DEFINITION 5.24. A symmetric bilinear form $B : L \times L \to k$ of a Lie algebra over a field k is said to be **invariant** if it satisfies

$$B([Y, X], Z) + B(X, [Y, Z]) = 0 \qquad (\forall X, Y, Z \in L)$$

(which means that "the Lie derivative of B is zero"), or, equivalently,

$$B([X, Y], Z) = B(X, [Y, Z]) \qquad (\forall X, Y, Z \in L)$$

(which means that B is "balanced".)

LEMMA 5.25. *Let L be a Lie algebra over a field k. Let B be an invariant bilinear form on L. Then for any ideal I of L,*

$$L^{\perp} = \{x \in L; B(x, y) = 0 (\forall y \in l)\}$$

is an ideal of L.

PROOF. Easy. □

Note: We need to be a bit careful when we use the notation $\bullet^{\perp}$. It is safer to clarify the "container" (L) and bilinear form ρ. So the lemma above we should have written $L^{\perp_{\rho,L}}$ (eek) in stead of $L^{\perp}$.

DEFINITION 5.26. Let (ρ, V) be a finite dimensional representation of a Lie algebra L over a field k. Then the **Killing form with respect to** (ρ, V) is a bilinear form on L defined by

$$\mathrm{Tr}_{\rho,V}(XY) = \mathrm{tr}_V(\rho(X)\rho(Y)).$$

The ordinary(usual) Killing formκ_L of L is a bilinear form on L defined as the Killing form of the adjoint representation. That is,

$$\kappa_L(X, Y) = \mathrm{Tr}_{\mathrm{ad},L}(XY) = \mathrm{tr}_{\mathrm{ad},L}(\mathrm{ad}(X)\,\mathrm{ad}(Y)).$$

It is easy to verify that the Killing forms defined as above are invariant.

5.7.1. *functoriality of Killing forms.*

LEMMA 5.27. *Let L be a Lie algebra over a field k. Then the followings are true.*

(1) *Let V be a finite dimensional representation of L. Let W be a subrepresentation of V. Then we have*

$$\mathrm{Tr}_V(xy) = \mathrm{Tr}_W(xy) + \mathrm{Tr}_{V/W}(xy).$$

(2) *Let I be an ideal of L. Assume L is finite dimensional. Then we have*

$$\kappa_L(x, y) = \mathrm{Tr}_{\mathrm{ad},L}(xy) = \mathrm{Tr}_{\mathrm{ad},I}(xy) + \mathrm{Tr}_{\mathrm{ad},L/I}(xy) = \mathrm{Tr}_{\mathrm{ad},I}(xy) + \kappa_{L/I}(\bar{x}, \bar{y})$$

(where $\bar{\bullet}$ denotes the class of $\bullet$ in L/I.) In particular, for any $x, y \in I$, we have

$$\kappa_L(x, y) = \kappa_I(x, y)$$

PROOF. (2): We choose a basis $B = B_1 \coprod B_2$ of L such that B_2 forms a basis of I. Then $\bar{B}_1$ forms a basis of L/I. Under the basis B, $\mathrm{ad}(x)$ may be represented by a matrix

$$\mathrm{ad}_L(x) = \begin{pmatrix} \mathrm{ad}_{L/I}(\bar{x}) & * \\ 0 & \mathrm{ad}_I x \end{pmatrix}.$$

We obtain the result easily from this.
(1): may be proved in a same manner. □

5.8. **Theorem of Iwasawa.**

THEOREM 5.28. *Let L be a Lie algebra over a field k of characteristic $p \neq 0$. Then L has a finite dimensional faithful representation. More precisely, there exists a two-sided ideal I of the universal enveloping algebra $U(L)$ such that L acts faithfully on $U(L)/I$.*

Before proving the above theorem, we first prove the next lemma.

LEMMA 5.29. *Under the hypothesis of the theorem, for any $x \in L$, there exists a monic non constant polynomial $f_x(X) \in k[X]$ such that*

$$f_x(x) \in Z(U(L))$$

holds.

PROOF. Let us put $s = \dim(L)$. The linear transformation $\mathrm{ad}(x)$ on L is represented by a matrix of size s and has therefore its minimal polynomial m_x: Namely, m_x is a monic polynomial of degree no more than s such that

$$m_x(\mathrm{ad}(x)) = 0$$

holds. Let us divide $X, X^p, X^{p^2}, \dots X^{p^{s+1}}$ by $m_x(X)$.

$$X^{p^j} = m_x(X)q_j(X)+r_j(X) \qquad (\deg(r_j) < s) \qquad (j = 1, 2, 3, \dots, s+1)$$

Then $s+1$ polynomials $\{r_j(X)\}_{j=1}^{s+1}$ of degree $\leq s-1$ should be linearly dependent. That means, there exists a non trivial vector $(c_j) \in k^{s+1}$ such that

$$\sum_j c_j X^{p^j} \in m_x(X)k[X]$$

holds. Then we have

$$\sum_j c_j(\mathrm{ad}(x))^{p^j} = 0.$$

Thus we conclude

$$\mathrm{ad}(\sum_j c_j x^{p^j}) = 0.$$

By dividing $\sum_j c_j X^{p^j}$ by leading coefficient, we obtain the required polynomial f_x.

□

PROOF. of the Theorem Let $\{e_1, e_2, \dots, e_s\}$ be a basis of L. Then by the above lemma we know that there exists a set of monic non constant polynomials $\{f_1, f_2, \dots, f_s\}$ such that each $h_j = f_j(e_j)$ belongs to the center of $U(L)$. Let us put $d_j = \deg(f_j)$. Then using PWB theorem we may easily see that

$$\left\{ h_1^{c_1} h_2^{c_2} h_3^{c_3} \dots h_s^{c_s} e_1^{l_1} e_2^{l_2} e_3^{l_3} \dots e_s^{l_s};\ \begin{array}{l} c_1, c_2, c_3, \dots, c_s \in \mathbb{N}, \\ l_1, l_2, l_3, \dots, l_s \in \mathbb{N}, \\ l_j < d_j (\forall j) \end{array} \right\}$$

forms a basis of $U(L)$. Let us now put

$$I = U(L)(h_1, \dots, h_s)$$

Then $A = U(L)/I$ is a finite dimensional vector space with the base

$$\left\{ e_1^{l_1} e_2^{l_2} e_3^{l_3} \dots e_s^{l_s};\ \begin{array}{l} l_1, l_2, l_3, \dots, l_s \in \mathbb{N}, \\ l_j < d_j (\forall j) \end{array} \right\}$$

The representation ρ_A of L on A is faithful. Indeed, for any $x \in L$, we have

$$\rho_A(x) = 0 \implies x.1 = 0 \text{ in } A \implies x \in I \implies x = 0.$$

□

DEFINITION 5.30. Let L be a Lie algebra.

(1) A representation V of L is called **completely reducible** if it is a direct sum of reducible sub representations.
(2) L is called **completely reducible** if every representation of L is completely reducible.

The following remark is (at least) in the Book of Bourbaki.

PROPOSITION 5.31. *Let L be a non zero finite dimensional Lie algebra over a field k of characteristic $p \neq 0$. Then L can never be completely reducible.*

PROOF. Let us follow the proof of the theorem of Iwasawa. By taking f_1^2 instead of f_1 in the proof, we obtain a representation $A_1 = U(L)/I$ with a non trivial central nilpotent $z = f_1(e_1)$. Then we see that zA_1 cannot have a direct complementary L-module X. For if it existed, then X should necessarily a left ideal of A_1. On the other hand, by decomposing $1 \in A_1$ we obtain

$$1 = x + za \qquad (\exists x \in X \exists a \in A_1).$$

Then $x = 1 - za$ has an inverse $(1 + za)$. This implies that

$$X \supset A_1 x \supset A_1.$$

which is a contradiction. □

5.9. **Cartan's criterion for solvability(Ccs).** Cartan's criterion relates several properties (semi simplicity, solvability) of Lie algebras with properties of their invariant bilinear forms.

To prove it we need some study on bilinear forms.

DEFINITION 5.32 (in this subsection only). For any free module R^t over a ring R, We denote by $\langle\rangle_R$ the "usual inner product". That is,

$$\langle v, w\rangle_R = \sum_i v_i w_i.$$

The first thing we do is to observe the property of this inner product when the base ring R is a "real field". (Since we only need it for the case $R = \mathbb{Q}$, we omit the definition of a real field and describe the following lemma only when $R = \mathbb{Q}$.)

LEMMA 5.33. *Let $W = \mathbb{Q}^t$ be a vector space. Let $b_1, b_2, \dots b_s \in W$. Let B be a $s \times s$ matrix defined by*

$$B = (\langle b_i, b_j\rangle_{\mathbb{Q}}).$$

Then the determinant of B is equal to 0 if and only if $\{b_j\}_{j=1}^s$ are linearly dependent over $\mathbb{Q}$.

PROOF. Assume $\{b_j\}_{j=1}^s$ are linearly dependent over $\mathbb{Q}$. Then there exists a non trivial vector $(c_1, c_2, \dots c_s) \in \mathbb{Q}^s$ such that

$$\sum_{j=1}^s c_j b_j = 0$$

holds. Thus

$$(c_1, c_2 \dots, c_s)B = 0$$

So B is a degenerate matrix which implies that $\det(B) = 0$.

Let us now prove the opposite implication. Assume $\det(B) = 0$. Then there exists a non trivial vector $(c_1, c_2, \dots, c_s)$ such that

$$(c_1, c_2 \dots, c_s)B = 0$$

holds. Let us put

$$v = \sum_j c_j b_j.$$

Then we see that $\langle v, v\rangle_{\mathbb{Q}} = 0$ and hence $v = 0$. (Note that for this implication we have used the fact that $\mathbb{Q}$ is a "real field".) Thus $\{b_j\}$ are linearly dependent over $\mathbb{Q}$. □

The next task is to compare $\mathbb{Q}$ with other field.

Definition 5.34. For any subset S of a $\mathbb{Z}$-module $W_{\mathbb{Z}}$, Let us put

$$\mathrm{MG}_S = \max\{|\det_{l\times l}(\langle b_i, b_j\rangle_{\mathbb{Z}})| b_1, \dots b_l \in S\}.$$

("The maximum modulus of Gram determinants".) We denote by S_k the subset of $W_k = W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \otimes k$ defined by

$$S_k = \{x \otimes 1 \in W \otimes_{\mathbb{Z}} k; x \in S\}.$$

Lemma 5.35. *Let S be a finite subset of a free module*

$$W_{\mathbb{Z}} = \mathbb{Z}^t = \{(v_1, v_2, \dots, v_t); v_j \in \mathbb{Z}(\forall j)\}$$

over $\mathbb{Z}$. Let k be a field of characteristic p. We assume either $p = 0$ or $p > \mathrm{MG}_S$ holds. Then we have

$$(a \in W_k, (a^\perp \cap S_k)^{\perp\perp} \ni a) \implies a = 0$$

Proof. Assume $a \neq 0$. Since the inner product $\langle \bullet, \bullet\rangle_k$ is non degenerate on W_k, we see that $(a^\perp \cap S_k)^{\perp\perp}$ is equal to the k-vector space spanned by $(a^\perp \cap S_k)$. Thus there exists a set of linearly independent vectors $\{b_j\} \subset S_k$ so that we may write down a as

$$a = \sum_i a_i b_i \qquad (a_i \in k).$$

Then by the assumption on a, we see that

$$\sum a_i \langle b_i, b_j\rangle_k = 0 \quad (\forall j)$$

Thus

$$\det(\langle b_i, b_j\rangle_k) = 0$$

which is equivalent to

$$p | \det(\langle b_i, b_j\rangle_{\mathbb{Z}}). \tag{1}$$

Note on the other hand that b_j are linearly independent over $\mathbb{Z}$. Thus

$$\det(\langle b_i, b_j\rangle_{\mathbb{Z}}) \neq 0$$

By the definition of MG_S, we see that

$$0 < |\det(\langle b_i, b_j\rangle_{\mathbb{Z}})| \leq \mathrm{MG}_S \tag{2}$$

which contradicts to the condition (1). □

Definition 5.36. For any positive integer n, and for any ring k, we denote by $\mathrm{Diag}_n(k)$ the set of diagonal matrices in $M_n(k)$. For any vector $a = (a_i) \in k^n$, we denote by $\mathrm{diag}(a)$ the diagonal matrix $\mathrm{diag}(a) = \mathrm{diag}(a_1, \dots, a_n)$. Note that for any ring k, the restriction of the Killing form of $\mathfrak{gl}_n$ coincides with the "usual" inner product with this identification. That is,

$$\mathrm{tr}(\mathrm{diag}(a)\,\mathrm{diag}(b)) = \langle a, b\rangle_k.$$

We define the following subset of $\mathrm{Diag}_n(\mathbb{Z})$.

$$S^{[n]} = \{(\mathrm{diag}((e_i - e_j) - (e_m - e_l)); i, j, m, l \in \{1, 2, 3, \dots, n\}\}.$$

(where the vectors $\{e_i\}_{i=1}^n$ are elementary vectors.) We note that an obvious estimate

$$\mathrm{MG}_{S^{[n]}} \leq 4^n n!$$

holds.

LEMMA 5.37. *Let n be a positive integer. Let $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in k^n$.*

If b satisfies $b \in (a^\perp \cap S^{[n]})^\perp$, then there exist polynomials $f, g \in k[X]$ such that

$$f(\mathrm{diag}(a)) = \mathrm{diag}(b), \quad g(\mathrm{ad}(\mathrm{diag}(a))) = \mathrm{ad}(\mathrm{diag}(b))$$

holds.

PROOF. Let us denote by ϵ_{ijlm} the vector

$$\epsilon_{ijlm} = (e_i - e_j) - (e_l - e_m).$$

We first find a map f_0 from $\Lambda_a = \{a_i; i = 1, 2, \dots, n\}$ to $\Lambda_b = \{b_i; i = 1, 2, \dots, n\}$ such that

$$f_0(a_i) = b_i.$$

Such a thing exists (is "well defined") if and only if

$$\forall i \forall j (a_i = a_j \implies b_i = b_j)$$

holds. This condition is equivalent to the condition

$$\forall i \forall j (a \perp \epsilon_{ij11} \implies b \perp \epsilon_{ij11})$$

which holds by the assumption on b. Thus we see that f_0 exists. On the other hand, by using Lagrange interpolation formula we see that there exists a polynomial $f \in k[X]$ such that $f|_{\Lambda_a} = f_0$. Then we have

$$f(\mathrm{diag}(a)) = \mathrm{diag}(b).$$

The adjoint action of a diagonal matrix $\mathrm{diag}(a)$ is represented by a diagonal matrix $(a_i - a_j)_{i,j}$. Thus an argument similar to the one above proves the existence of g.

□

PROPOSITION 5.38 (Cartan). *Let V be an n dimensional vector space over a field k of characteristic p. We assume that either $p = 0$ or $p > \mathrm{MG}_{S^{[n]}}$ holds. Let L be a Lie subalgebra of $\mathfrak{gl}(V)$. If the Killing form of L with respect to V is identically equal to 0, then L is solvable.*

Proof. We may assume that k is algebraically closed. Let us take an element $x \in [L, L]$. Then we have

$$(\operatorname{ad} x_s)(L) = (\operatorname{ad} x)_s(L) \subset L$$

Let us now diagonalize x_s and write $x_s = \operatorname{diag}(a)$. Let us take arbitrary $b \in (a^\perp \cap S^{[n]})^\perp$. By the lemma above we see that there exist polynomials $f, g \in k[X]$ such that

$$f(\operatorname{diag}(a)) = \operatorname{diag}(b), \quad g(\operatorname{ad}(\operatorname{diag}(a))) = \operatorname{ad}(\operatorname{diag}(b))$$

holds. For any $w = \sum_l [y_l z_l] \in [L, L]$, we have:

$$\begin{aligned}
&\operatorname{tr}(\operatorname{diag}(b)[\sum_l y_l z_l]) = \sum_l \operatorname{tr}([\operatorname{diag}(b), y_l] z_l) \\
=&\sum_l \operatorname{tr}(\operatorname{ad}(\operatorname{diag}(b)).y_l\ z_l) = \sum_l \operatorname{tr}(g(\operatorname{ad}(\operatorname{diag}(a))).y_l\ z_l) \\
&= \sum_l \operatorname{tr}(g(\operatorname{ad}(x_s)).y_l\ z_l) \in \sum_l \operatorname{tr}(LL) \\
&= 0
\end{aligned}$$

That means, $\operatorname{tr}(\operatorname{diag}(b)w) = 0$. In particular, we have

$$\operatorname{tr}(\operatorname{diag}(b)x) = 0.$$

Since $\operatorname{diag}(b) = f(\operatorname{diag}(a)) = f(x_s)$ is a polynomial in x, it commutes with x_s and with x_n. thus

$$\operatorname{diag}(b)x = (\operatorname{diag}(b)x_s) + (\operatorname{diag}(b)x_n)$$

gives the Jordan-Chevalley decomposition of $\operatorname{diag}(b)x$. Therefore,

$$0 = \operatorname{tr}(\operatorname{diag}(b)x) = \operatorname{tr}(\operatorname{diag}(b)x_s) = \operatorname{tr}(\operatorname{diag}(b)\operatorname{diag}(a)) = \langle b, a\rangle_k$$

thus $b \perp a$.

To sum up, we have shown

$$(a^\perp \cap S)^\perp \ni b \implies b \in a^\perp.$$

In other words,

$$(a^\perp \cap S)^{\perp\perp} \ni a$$

which is equivalent to saying that a is a linear combination of elements in $(a^\perp \cap S)$.

In view of Lemma 5.35, we see that $a = 0$. So $x = x_s + x_n = x_n$ is a nilpotent element.

By the theorem of Engel, we conclude that $[L, L]$ is nilpotent. Thus L is solvable (since we have shown that $L/[L, L]$ and $[L, L]$ are solvable). $\square$

Definition 5.39. We say that the Cartan's criterion for solvability (Ccs) holds for a linear Lie algebra $L \subset (\mathfrak{gl}_n(k))$ over a field k if it satisfies the following condition.
(Ccs) If the Killing form on L associated to k^n is identically zero, L is solvable.

Let n be a positive integer. We denote by $\mathrm{Ccs}(n)$ the set of p such that Ccs holds for any Lie algebra L of dimension less than or equal to n for any field k of characteristic p.

$$\mathrm{Ccs}(n) = \{p; \text{Ccs holds for any}(L \subset \mathfrak{gl}_n(k)) \text{ provided } \operatorname{char}(k) = p, \}$$

Corollary 5.40 (of Proposition). *Let n be a positive integer. Then:*

(1) $0 \in \mathrm{Ccs}(n)$
(2) *For any prime p which is larger than $\mathrm{MG}_{S[n]}$, we have $p \in \mathrm{Ccs}(n)$.*
(3) *In particular, for any prime p which is larger than $4^n n!$, we have $p \in \mathrm{Ccs}(n)$.*

Note:

The estimate given in the above corollary is presumably far from the best one.

Proposition 5.41. *Let n be a positive integer. Let k be a field of characteristic $p \in \mathrm{Ccs}(n)$. Let L be a Lie algebra over k whose dimension is less than or equal to n. If the usual Killing form $\kappa = \mathrm{Tr}_{\mathrm{ad},L}$ of L is identically equal to zero, then L is solvable. In particular, if $p > 4^n n!$ or $p = 0$, then L is solvable if its usual Killing form is identically equal to zero.*

Proof. Apply the definition to

$$L/(\text{center of} L) \hookrightarrow \mathfrak{gl}_n(L).$$

□

5.10. Cartan's criterion for semisimplicity.

Definition 5.42. We call a Lie algebra L over k **non degenerate** if the Killing form κ_L of L is non degenerate.

Lemma 5.43. *Every non degenerate Lie algebra L over a field k is semisimple.*

Proof. Assume that there exists a non trivial abelian ideal A of L. Let y_0 be a non zero element of A. Then for any $x \in L$, $z = \mathrm{ad}(x)\,\mathrm{ad}(y_0)$ is nilpotent. Indeed,

$$z(L) = \mathrm{ad}(x)\,\mathrm{ad}(y_0)(L) = \mathrm{ad}(x)([y_0, L]) \subset \mathrm{ad}(x)(A) \subset A,$$

$$z^2(L) = \mathrm{ad}(x)\,\mathrm{ad}(y_0)(z(L)) \subset \mathrm{ad}(x)\,\mathrm{ad}(y_0)(A) = \mathrm{ad}(x)[y_0, A] = 0.$$

Thus $\kappa(x, y_0) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y_0)) = 0$ for for any $x \in L$. That means, $y_0 \in L^\perp$. This is contrary to the assumption on L. □

PROPOSITION 5.44. *Let n be a positive integer. Let k be a field of characteristic $p \in \mathrm{Ccs}(n)$. Let L be a Lie algebra of dimension n. Then the following conditions are equivalent:*

(1) *L is semisimple.*
(2) *L is non degenerate.*
(3) *L is a direct sum of simple ideals.*

PROOF. ((1) $\implies$ (2)): Assume L is semisimple. Let us take an ideal $I = L^\perp$ of L. Then the Killing form on I is identically equal to zero. since $\dim(I) \leq n$, I is a solvable algebra. Since L is semisimple, this implies $I = 0$.
((2) $\implies$ (1)): holds (regardless of the base field) in view of the previous lemma.
((3) $\implies$ (2)): We see that simple algebras are non degenerate in view of the argument above. Thus L is also non degenerate.
((2) $\implies$ (3)): Let H be a nontrivial ideal of L. Then $H \cap H^\perp$ is an abelian ideal of L. Indeed, for any $y, z \in H \cap H^\perp$ and for any $x \in L$, we have

$$\kappa(x, [y, z]) = \kappa([x, y], z) \in \kappa(H, H^\perp) = 0$$

So that $[y, x] \in L^\perp = 0$. On the other hand, by the previous lemma we see that $H \cap H^\perp$ is semisimple and so we have $H \cap H^\perp = 0$. Accordingly we have $L = H \oplus H^\perp$.

□

5.11. **examples.**

EXAMPLE 5.45. Let k be a field of characteristic p (possibly 0).

$$\mathfrak{gl}_n(k)$$

is a Lie algebra with the Killing form

$$\begin{aligned}
\kappa_{\mathfrak{gl}_n(k)}(x, y) &= \mathrm{tr}((\lambda(x) - \rho(x))(\lambda(y) - \rho(y))) \\
&= \mathrm{tr}(\lambda(xy)) + \mathrm{tr}(\rho(xy)) - \mathrm{tr}(\lambda(x)\rho(y) - \mathrm{tr}(\lambda(y)\rho(x)) \\
&= 2n\,\mathrm{tr}(xy) - 2\,\mathrm{tr}(x)\,\mathrm{tr}(y).
\end{aligned}$$

$\mathfrak{sl}_n(k)$ is an ideal of $\mathfrak{gl}_n(k)$ and so its Killing form is given by

$$\kappa_{\mathfrak{sl}_n(k)}(x, y) = 2n\,\mathrm{tr}_{k^n}(xy).$$

If $p \not| 2n$, then the Killing form is easily seen to be non-degenerate. so $\mathfrak{sl}_n(k)$ is a non-degenerated Lie algebra in this case. In this way we see that it is a semisimple Lie algebra. The Lie algebra is actually simple as we have shown in Proposition 5.22.

Example 5.46. Let p be an odd prime. Let k be a field of characteristic p. Then we have shown in Proposition 5.19 that the only non trivial ideals of $\mathfrak{gl}_p(k)$ are $\mathfrak{sl}_p(k)$ and $k.1_p$. So we see that

$$L = \mathfrak{gl}_p(k)/k.1_p$$

is a semisimple Lie algebra (as it has no proper abelian ideals). It has a unique nontrivial ideal

$$M = \mathfrak{sl}_p(k)/k.1_p.$$

Thus L cannot be a direct sum of simple Lie algebras.

5.12. **Weyl's theorem on complete reducibility.**

Definition 5.47. Let L be a finite dimensional Lie algebra. Let B be a non-degenerate invariant bilinear form on L. Then we define the **Casimir element** $C_B \in U(L)$ with respect to B by

$$C_B = \sum_i x_i x^{(i)}$$

where $\{x_i\}$ is a basis of L, and $\{x^{(i)}\}$ is the dual basis of the basis $\{x_i\}$ with respect to B.

Proposition 5.48. *Under the same assumption of the definition above, we have the following facts.*

1. *The Casimir operator C_B is independent of the choice of the basis $\{x_i\}$ of L.*
2. *C_B commutes with L. So it is in the center of $U(L)$.*

Proof. (1): easy exercise in linear algebra.
(2): For any $a \in L$, let us write the adjoint action of a on L by using the basis $\{x_i\}$. Namely,

$$[a, x_i] = \sum_j c_i^{(j)}(a) x_j \qquad (c_i^{(j)}(a) \in k).$$

Then the constants $\{c_i^{(l)}(a)\}$ ("structure constants") are expressed in terms of B as follows.

$$B(x^{(l)}, [a, x_i]) = \sum_j c_i^{(j)}(a) B(x^{(l)}, x_j) = c_i^{(l)}(a)$$

We note that from the invariance of B, we have

$$B([x^{(l)}, a], x_i) = c_i^{(l)}(a),$$

so that we have a dual expression

$$[x^{(l)}, a] = \sum_i c_i^{(l)}(a) x^{(i)}.$$

Then we compute as follows.

$$\begin{aligned}[a, C_B] &= \sum_i [a, x_i] x^{(i)} + \sum_i x_i [a, x^{(i)}] \\ &= \sum_i \sum_j c_i^{(j)}(a) x_j x^{(i)} - \sum_i \sum_j c_j^{(i)}(a) x_i x^{(j)} = 0.\end{aligned}$$

□

Definition 5.49. Let L be a finite dimensional Lie algebra. Let V be a finite dimensional L-module. $\mathrm{Tr}_V(\bullet\bullet)$ with respect to V. We assume that the Killing form $\mathrm{Tr}_V(\bullet\bullet)$ with respect to V is non degenerate. Then we define the **Casimir element** with respect to V by

$$C_V = C_{\mathrm{Tr}_V}$$

where $\{x_i\}$ is a basis of L, and $\{x^i\}$ is the dual basis of the basis $\{x_i\}$ with respect to the Killing form Tr_V.

Lemma 5.50. *Let k be a field of characteristic p. Let L be a n-dimensional semisimple Lie algebra over a field k. Let V be a m-dimensional L-module. Let I be the kernel of the representation ρ_V associated to V. We assume $p \in \mathrm{Ccs}(n) \cap \mathrm{Ccs}(m)$. Then the Killing form Tr_V on L/I is non degenerate.*

Proof. L is semisimple and $p \in \mathrm{Ccs}(n)$ so L is semisimple. L/I is also non-degenerate so L/I is semisimple. We may thus assume $I = 0$. An ideal

$$J = L^{\perp_{\mathrm{Tr}_V}}$$

of L is a solvable ideal. Since L is semisimple and $p \in \mathrm{Ccs}(m)$, we have by $J = 0$. That means, Tr_V is non-degenerate on L.

□

Lemma 5.51. *Let k be a field of characteristic p (which may be 0). Let $(L, W \subset V)$ be a triple which satisfies the following conditions.*

1. *L is a semisimple Lie algebra over k.*
2. *V is a finite dimensional irreducible L-module.*
3. *W is an L-submodule of V of codimension 1.*
4. *$p \in \mathrm{Ccs}(\dim(L)) \cap \mathrm{Ccs}(\dim(V))$.*

Then the exact sequence

$$0 \to W \to V \to V/W \to 0$$

splits. In other words, there exists a 1*-dimensional* L*-submodule* X *of* V *which is complementary to* W*.*

PROOF. Since the question of existence of X is described in terms of existence of a solution of a set of linear equations, we may assume that k is algebraically closed. Let us denote by ρ_V the representation of L associated to V. Then by replacing L by $L/\ker(\rho_V)$ if necessary, we may assume that the representation ρ_V is faithful.

Note that since L is semisimple, it acts on V/W trivially.

Let us first treat the case where W is irreducible. Let $c = c_V$ be a Casimir element with respect to V. Since L is acts on V/W trivially, $c|_{V/W}$ is equal to zero. Thus

$$\dim(V) = \mathrm{tr}_V(c) = \mathrm{tr}_W(c|_W) + \mathrm{tr}_{V/W}(c|_{V/W}) = \mathrm{tr}_W(c|_W).$$

In particular, $c|_W$ is not equal to zero. On the other hand, by Schur's lemma, $c|_W$ is equal to a scalar $\lambda \in k$. Thus $X = \mathrm{Ker}(c)$ is a required object in this case.

We next come to general case. Let W_1 be the maximal proper L-submodule of W. Then we see that $(L, W/W_1 \subset V/W_1)$ satisfies the assumption of the lemma with W/W_1 irreducible. By the argument above, we therefore see that there exists an L-submodule Y which contains W_1 as a submodule of codimension 1 such that

$$V/W_1 = Y/W_1 \oplus W/W_1$$

holds. Since $(L, W_1 \subset Y)$ also satisfies the assumption of the lemma with $\dim(Y) < \dim(V)$, we deduce by induction that the lemma holds in general.

□

LEMMA 5.52. *Let* L *be a Lie algebra over a commutative ring* k*. Then for any* L*-modules* V, W*, each of the vector spaces*

$$\mathrm{Hom}_{k\text{-linear}}(V, W)$$

and

$$V \otimes_k W$$

admits a structure of L*-module. Namely,*

$$(x.f)(v) = x.(f(v)) - f(x.v) \qquad (\forall x \in L, \forall f \in \mathrm{Hom}_{k\text{-linear}}(V, W) \forall v \in V),$$

$$(x.v \otimes_k w) = (x.v) \otimes w + v \otimes_k (x.w) \qquad (\forall x \in L, \forall v \in V, \forall w \in W).$$

PROOF. Easy. □

Theorem 5.53 (Weyl). *Let k be a field of characteristic p (which may be 0). Let L be a non degenerate Lie algebra over k. Let V be a finite dimensional irreducible L-module. Assume $p \in \mathrm{Ccs}(\dim(V)^2)$. Then V is completely reducible.*

Proof. Let us define the following L-modules.

$$V_1 = \{f \in \mathrm{Hom}_{k\text{-linear}}(V, W); f|_W \in k.1_W\}$$
$$W_1 = \{f \in \mathrm{Hom}_{k\text{-linear}}(V, W); f|_W = 0\}$$

Then it is easy to see that the triple $(L, W_1 \subset V_1)$ satisfies the assumption of Lemma 5.51. We therefore have an element $f \in \mathrm{Hom}_{k\text{-linear}}(V, W)$ which satisfies the following conditions.

(1) $f|_W \in k.1_W$.
(2) $f|_W \neq 0$.
(3) $x.f = 0 \quad (\forall x \in L)$ (In other words, f is a L-linear homomorphism).

by dividing by a suitable element in k, we may assume $f|_W = 1_W$. Then f gives a splitting of the exact sequence

$$0 \to W \to V \to V/W \to 0.$$

□

5.13. **Semi direct products of Lie algebras.**

Definition 5.54. Let L be a Lie algebra over a commutative ring k. Then:

(1) A (k-linear) **derivation** of L is a k-linear map $D : L \to L$ such that it obeys the following "Leibniz rule".
$$D([x, y]) = [Dx, y] + [x, Dy] \qquad (\forall x, y \in L).$$
(2) We denote by $\mathrm{Der}_k(L)$ the set of all derivations of L.

Lemma 5.55. (1) *Any derivation D of a Lie algebra L is lifted to a derivation on the universal enveloping algebra $U(L)$.*
(2) *$\mathrm{Der}_k(L)$ forms a Lie algebra under the usual k-linear structure and the usual commutator as the bracket product.*

Definition 5.56. Let L_1, L_2 be Lie algebras over a commutative ring k. we say "L_1 **acts on** L_2 **as a derivation**" if there is given a Lie algebra homomorphism

$$\pi : L_1 \to \mathrm{Der}_k(L_2).$$

If the action π is obvious in context, we shall simply denote $x.y$ instead of $\pi(x).y$.

Definition 5.57. Let L_1, L_2 be Lie algebras over a commutative ring k. Assume there is given an action π of L_1 on L_2. Then we define a **semi direct product** $L_1 \ltimes_\pi L_2$ of L_1 and L_2 by introducing the k-module $L_1 \oplus L_2$ with the following bracket product.

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2] + x_1.y_2 - y_1.x_2).$$

Note that

(1) L_1 and L_2 are (identified with) subalgebras of $L_1 \ltimes_\pi L_2$.
(2) Further more, L_2 is an ideal of $L_1 \ltimes_\pi L_2$.
(3) For $x \in L_1$ and $y \in L_2$, we have
$$[x, y]_{L_1 \ltimes L_2} = x.y.$$

5.14. Levi decomposition.

Definition 5.58. Let L be a Lie algebra over a field k. Let R be the radical of L. A **Levi-subalgebra** of L is a subalgebra L_1 of L such that L is a direct sum of L_1 and R as a vector space over k.

We have the following obvious lemma.

Lemma 5.59. *Let L be a Lie algebra over a field k. Let R be the radical of L. Let L_1 be a Levi-subalgebra of L. Then:*

(1) $L \cong L_1 \ltimes R$.
(2) $L_1 \cong L/R$.

In particular the isomorphism class of L_1 is unique.

□

Lemma 5.60. *Let n be a positive integer. Let L be an n-dimensional Lie algebra over a field k of characteristic $p \in \mathrm{Ccs}(n^2)$. Assume the radical R of L is abelian. Then:*

(1) $R = L^\perp$. *(We equip L with the usual Killing form.)*
(2)
$$\mathrm{Hom}_{k\text{-linear}}(L, R)$$
admits an action α of L. Namely,
$$(\alpha(x).\varphi)(y) = [x, \varphi(y)] - \varphi([x, y]) \qquad (x \in L, y \in L).$$
(3) *For any $x \in L$, we have $(\alpha(x).\varphi))|_R = 0$.*
(4) *For any $x \in R$, $\alpha(x)$ is nilpotent.*
(5)
$$V_1 = \{\varphi \in \mathrm{Hom}_{k\text{-linear}}(L, R); \varphi|_R \in k \cdot \mathrm{id}_R\}$$
is an R-submodule of $\mathrm{Hom}_{k\text{-linear}}(L, R)$.
(6) *If R is not equal to zero, then*
$$V_2 = \{\varphi \in \mathrm{Hom}_{R\text{-module}}(L, R); \varphi|_R \in k \cdot \mathrm{id}_R\} \neq 0.$$

(7) *There exists a Levi subalgebra L_1 of L.*

PROOF. (1) Since $L^{\perp}$ has a trivial Killing form, it is solvable. (Cartan's criterion.) Thus, by the maximality of R, we have

$$R \supset L^{\perp}.$$

On the other hand, let us take an arbitrary $x \in R$, then for any $y, z \in L$, we have

$$\mathrm{ad}(x)\,\mathrm{ad}(y)z = [x,[y,z]] \in [x,R] \subset R,$$

$$(\mathrm{ad}(x)\,\mathrm{ad}(y))^2 z \in [x,[y,R]] \subset [x,R] = 0 \quad \text{(Since } R \text{ is abelian.)}$$

So $\kappa_L(x,y) = \mathrm{tr}_L(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$ (as a trace of a nilpotent element.) Thus

$$R \subset L^{\perp}$$

also holds.
(2):follows from the general theory.
(3):follows easily from the definition of α.
(4):Let us take an arbitrary $x \in R$. For any $y \in L$, we have by using(2)

$$((\alpha(x)^2).\varphi)(y) = [x,(x.\varphi)(y)] - (x.\varphi)([x,y]) \in [R,R] - (x.\varphi)(R) = 0.$$

Thus $\alpha(x)^2 = 0$.
(5): follows clearly from (3).
(6): follows from (4),(5) and Engel's theorem. (We need to note that

$$V_2 = \{\varphi \in V_1; \forall x \in R(\alpha(x).\varphi = 0)\}$$

holds.) (7): The action of R on V_2 is equal to zero. So V_2 admits an action by L/R, which is semisimple. Now consider the following exact sequence of L/R-modules.

$$0 \to V_0 \to V_2 \xrightarrow{|_R} k \to 0$$

where V_0 is the kernel of the restriction map. By a special case of Weyl's theorem on complete reducibility (Lemma 5.51), We see that the sequence splits. (Since we assumed $p \in \mathrm{Ccs}(n^2)$). This implies that there exists an element $\varphi_0 \in V_2$ such that

$$\alpha(x).\varphi_0 = 0, \qquad \varphi_0|_R = \mathrm{id}_R\,.$$

Thus φ_0 gives a splitting of the injection $R \subset L$. A Levi subalgebra of L is obtained by putting

$$L_1 = \{x - \varphi_0(x); x \in L\}.$$

□

Theorem 5.61 (Levi decomposition of a Lie algebra). *Let n be a positive integer. Let $p \in \mathrm{Ccs}(n^2)$. Let L be a n-dimensional Lie algebra over a field of characteristic p. Then L has a Levi subalgebra L_1. In other words, L may be expressed as a semi direct product*

$$L = L_1 \ltimes R$$

where L_1 is a semisimple (Levi) subalgebra of L, and R is a solvable (radical) ideal of L.

Proof. If $R = 0$, then we only need to set $L_1 = L$. So let us assume $R \neq 0$. Let us put

$$R_1 = [R, R].$$

Then from the definition, we R/R_1 is an abelian Lie algebra. It is also easy to verify that R_1 is an ideal of L. (R_1 is a **characteristic ideal** of R). We apply the preceding lemma for $R/R_1 \subset L/R_1$ to obtain a Levi subalgebra M/R_1 of L/R_1. Then M satisfies the following relations.

$$M + R = L, \quad M \cap R = R_1.$$

Since R is solvable (and we have assumed $R \neq 0$), we see that $\dim(M)$ is strictly smaller than $\dim(L)$. By induction M have a Levi subalgebra M_1. Then it is clear that M_1 is a Levi subalgebra of L.

□

5.15. **Abstract Jordan Chevalley decomposition.**

Proposition 5.62. *Let n be a positive integer. Let L be a n-dimensional semisimple Lie algebra over a field k of characteristic $p \in \mathrm{Ccs}(n^4)$. Then any derivation $D \in \mathrm{Der}_k(L)$ of L is inner. That is, there exists an element $x = x_D$ such that*

$$D(y) = [x_D, y] = \mathrm{ad}(x_D)(y).$$

Proof. $\mathrm{Der}_k(L)$ is itself a Lie algebra. Sending each element x of L to its "inner derivation" $\mathrm{ad}(x)$, we obtain a Lie algebra homomorphism

$$\mathrm{ad} : L \to \mathrm{Der}_k(L)$$

We note that $\dim(\mathrm{Der}_k(L)) \leq \dim(L)^2$, and that ad may be viewed as a homomorphism of L-modules. (L acts on $\mathrm{Der}_k(L)$ via ad. Namely,

$$x.D = \mathrm{ad}(x).D = [\mathrm{ad}(x), D] = [x, D\bullet] - D([x, \bullet]) = -\mathrm{ad}(D.x)$$

holds for any $x \in L$ and for any $D \in \mathrm{Der}_k(L)$.) By the Weyl's theorem on complete reducibility, we see that there exists a direct sum decomposition

$$\mathrm{Der}_k(L) = \mathrm{ad}(L) \oplus X$$

of L-modules. Then for any $D \in X$ and for any $x \in L$, we see that

$$x.D(= -\operatorname{ad}(D.x)) \in X \cap \operatorname{ad}(L) = 0.$$

So $D = 0$. That means, $X = 0$.

□

PROPOSITION 5.63. *Let n be a positive number Let k be a separably closed field of characteristic $p \in \operatorname{Ccs}(n^4)$. We assume further that n is invertible in k. (This assumption is provided just in case: it probably is not necessary because the assumption $p \in \operatorname{Ccs}(n^4)$ is presumably much stronger.) Let $L \subset \mathfrak{gl}_n(k)$ be a linear semisimple Lie algebra. We assume that the representation $L \curvearrowright k^n$ is irreducible. Then for any element $x \in L$, its semisimple part x_s and its nilpotent part x_n in $\mathfrak{gl}_n(k)$ lies in L.*

PROOF. We may assume k is algebraically closed. Let $x \in L$ It is enough to prove $x_n \in L$. There exists a polynomial $f \in k[X]$ such that $x_n = f(x)$. Thus we see

$$\operatorname{ad} x_n(L) \subset L.$$

Thus $\operatorname{ad} x_n$ is a derivation of L. By the preceding lemma we see that there exists an element $y \in L$ such that

$$\operatorname{ad} x_n = \operatorname{ad} y$$

By Schur's lemma, we see that there exists a constant $c \in k$ such that

$$x_n = y + c \cdot 1_n.$$

Let us compute traces of both hand sides. Since $L = [L, L]$ (L has no non-trivial ideals.), we have $\operatorname{tr}(y) = 0$. Since x_n is nilpotent, we have $\operatorname{tr}(x_n) = 0$. Thus we conclude $c = 0$ (as we assumed n is invertible in k.) □

PROPOSITION 5.64. *Let n be a positive integer. Let L be a semisimple Lie algebra over a separably closed field k of characteristic $p \in \operatorname{Ccs}(n^4)$. Let $V_1 = (V_1, \pi_1), V_2 = (V_2, \pi_2)$ be faithful irreducible representations of L with dimensions less than or equal to n. Then for any $x \in L$, the Jordan Chevalley decomposition of x*

$$x = x_s^{(1)} + x_n^{(1)}$$

with respect to V_1 and that

$$x = x_s^{(2)} + x_n^{(2)}$$

with respect to V_2 coincides.

Proof. We consider a faithful representation $(V,\pi) = (V_1 \oplus V_2, \pi_1 \oplus \pi_2)$. For any $x \in L$,

$$\pi(x) = \begin{pmatrix} \pi_1(x) & 0 \\ 0 & \pi_2(x) \end{pmatrix} = \begin{pmatrix} \pi_1(x_s^{(1)}) & 0 \\ 0 & \pi_2(x_s^{(2)}) \end{pmatrix} + \begin{pmatrix} \pi_1(x_n^{(1)}) & 0 \\ 0 & \pi_2(x_n^{(2)}) \end{pmatrix}$$

satisfies the requirement for the Jordan Chevalley decomposition so by the uniqueness we see

$$\pi(x)_s = \begin{pmatrix} \pi_1(x_s^{(1)}) & 0 \\ 0 & \pi_2(x_s^{(2)}) \end{pmatrix}, \quad \pi(x)_n = \begin{pmatrix} \pi_1(x_n^{(1)}) & 0 \\ 0 & \pi_2(x_n^{(2)}) \end{pmatrix}.$$

Now we argue in a same way as in the proof of the previous proposition and see that there exists a unique element $y \in L$ such that

$$\mathrm{ad}(x_n) = \mathrm{ad}(y)$$

holds. By comparing entries, we obtain

$$\mathrm{ad}(\pi_1(y)) = \mathrm{ad}(\pi_1(x_n^{(1)})), \quad \mathrm{ad}(\pi_2(y)) = \mathrm{ad}(\pi_2(x_n^{(2)})).$$

Since L has trivial center, we have

$$\pi_1(y) = \pi_1(x_n^{(1)}), \quad \pi_2(y) = \pi_2(x_n^{(2)}).$$

Thus $y = x_n^{(1)} = x_n^{(2)}$. $\square$

Definition 5.65. Let n be a positive integer. Let L be an n-dimensional semisimple Lie algebra over a separably closed field k of characteristic $p \in \mathrm{Ccs}(n^4)$. Then the **abstract Jordan Chevalley decomposition** of x is an decomposition

$$x = x_s + x_n \qquad (x_s, x_n \in L)$$

such that

$$\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$$

is the Jordan Chevalley decomposition.

Proposition 5.66. *Let n be a positive integer. Let L be an n-dimensional Lie algebra over a separably closed field k of characteristic $p \in \mathrm{Ccs}(n^4)$ Then the abstract Jordan Chevalley decomposition of x exists. If furthermore there is given a m-dimensional representation (V,π) of L and $p \in \mathrm{Ccs}(m^4)$, then*

$$\pi(x) = \pi(x_s) + \pi(x_n)$$

gives the Jordan Chevalley decomposition of x.

Proof. Easy exercise. (Be sure to use Weyl's theorem of complete reducibility. By taking quotient by a certain ideals (kernels of representations) one may reduce the proposition to a case where L is semisimple and π is faithful and irreducible.)

□

References

[1] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, 1972.